# THE $D$-EQUIVALENCE CONJECTURE FOR MODULI SPACES OF SHEAVES

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In [BO], Bondal and Orlov made the following

**Conjecture 0.1** ($D$-equivalence)**.** *If $X$ and $Y$ are birationally equivalent projective Calabi-Yau manifolds, then there is an equivalence of derived categories of coherent sheaves* $\mathrm{DCoh}(X) \to \mathrm{DCoh}(Y)$.

This conjecture is motivated by homological mirror symmetry. Although it has been established in dimension three [B], and various "local" versions of the conjecture have been established, there are few examples known for compact Calabi-Yau manifolds in dimension $> 3$. The purpose of this note is to sketch a proof of the following

**Theorem 0.2.** *The $D$-equivalence conjecture holds for Calabi-Yau manifolds which are birationally equivalent to a moduli space of Gieseker semistable coherent sheaves (of some fixed primitive Mukai vector) on a $K3$ surface.*

This theorem will actually be a consequence of a more general theorem, Theorem 3.6 which leads to derived equivalences for any birational equivalence resulting from "variation of stability" on a derived stack with self-dual cotangent complex. The theorem above, and in fact a slightly more general statement, arises from applying this theorem to moduli spaces of Bridgeland semistable objects in the derived category of a twisted $K3$-surface as one varies the stability condition. The approach we present has several key new ingredients:

1. The theory of $\Theta$-stratifications, and the general structure theorem for the derived category of coherent complexes on a quasi-smooth stack with a $\Theta$-stratification, leading to the aforementioned Theorem 3.6 on derived equivalences;
2. A new theorem on the local structure of derived stacks with self-dual cotangent complex which admit a good moduli space, Theorem 2.3, powered by the Luna slice theorem of [AHR];
3. The minimal model program for Calabi-Yau manifolds which are birationally equivalent to a moduli space of coherent sheaves on a $K3$-surface [BM]; and
4. A new existence result for good moduli spaces of Bridgeland semistable objects in the derived category of a twisted $K3$-surface, Theorem 4.3.

This paper will ultimately be a section in a larger paper [HL5]. I have written this sketch of the argument now because I have announced these results in several lectures, and people have expressed interest in a written summary.

I call this a "sketch" of proof because there are certain points which depend on forthcoming work. Notably, we use the general structure theorem for the derived category of a derived stack with a $\Theta$-stratifications which was announced in [HL4]. This will appear in the final version of [HL5], but the special case of local quotient stacks already appears in [HL3]. We also use a forthcoming result of the author, Alper, and Heinloth on the existence of good moduli spaces, Theorem 4.3. Finally, our discussion of the moduli stacks of Bridgeland semistable twisted complexes on a $K3$-surface is somewhat telegraphic, and we will fill in some of the details of the discussion in the final version of this paper.

In sections 1,2, and 3, we work over an arbitrary base field $k$ of characteristic 0, and in section 4 we specialize to the case $k = \mathbb{C}$.

# Contents



## 1. A remark on intrinsic geometric invariant theory

A key technical tool will be an intrinsic version of the main theorem of geometric invariant theory for stacks $\mathcal{X}$ which admit a "good moduli space" in the sense of [A]. We also use the theory of $\Theta$-stability [HL1], which provides methods for constructing canonical stratifications of algebraic stacks. The stratification is defined intrinsically via a generalized version of the Hilbert-Mumford criterion formulated in terms of maps $\Theta \to \mathcal{X}$, where $\Theta := \mathbb{A}^1/\mathbb{G}_m$.

**Theorem 1.1.** *Let $\mathcal{X} \to Y$ be a good moduli space, where $\mathcal{X}$ is a finite type geometric stack.*[1] *Then an class $\ell \in NS(\mathcal{X})_\mathbb{R}$ and positive definite $b \in H^4(\mathcal{X};\mathbb{R})$ induces a $\Theta$-stratification*

$$\mathcal{X} = \mathcal{X}^{\mathrm{ss}}(\ell) \cup \mathcal{S}_0 \cup \cdots \cup \mathcal{S}_N.$$

*$\mathcal{X}^{\mathrm{ss}}(\ell)$ admits a good moduli space $\mathcal{X}^{ss}(\ell) \to Y'$, and $Y'$ is projective over $Y$. The formation of this stratification is étale local on $Y$.*

If $k'/k$ is a field extension, then a $k'$-point $p \in \mathcal{S}_\alpha(k')$ classifies a canonical map $f : \Theta_{k'} \to \mathcal{X}$ with $f(1) \simeq p$, which we refer to as the *Harder-Narasimhan (HN)* filtration of the $\ell$-unstable point $p$. The strata canonically deformation retract onto the "centers" $\mathcal{Z}^{\mathrm{ss}}_\alpha$, which classify maps $(B\mathbb{G}_m)_{k'} \to \mathcal{X}$. The projection $\pi : \mathcal{S}_\alpha \to \mathcal{Z}^{\mathrm{ss}}_\alpha$ maps a HN filtration $f : \Theta_{k'} \to \mathcal{X}$ to its "associated graded," the restriction of $f$ to a map $\{0\}/(\mathbb{G}_m)_{k'} \to \mathcal{X}$.

The key to the proof of theorem Theorem 1.1 is the final claim, that the construction is local on $Y$. The proof is an elementary consequence of the recent "Luna étale slice theorem" for stacks established in [AHR], one consequence of which is the following:

**Theorem 1.2.** *Let $\mathcal{X} \to Y$ be a good moduli space, where $\mathcal{X}$ is a finite type geometric stack. Then there is an étale surjection* $\mathrm{Spec}(R) \to Y$ *such that the base change of $\mathcal{X}$ to* $\mathrm{Spec}(R)$ *is a quotient of an affine scheme by a linearly reductive $k$-group.*

A more precise formulation states that if $\mathcal{X} \to Y$ is a good moduli space, then any point $y \in Y$ is contained in the image of some ëlate map $U \to Y$ such that $\mathcal{X} \times_Y U \simeq \mathrm{Spec}(R)/G$ where $G = \mathrm{Aut}(x)$ for some closed point $x \in \mathcal{X}$ in the fiber over $y \in Y$. Theorem 1.2 allows one to reduce Theorem 1.1 to the classical situation of an affine scheme modulo a linearly reductive group.

## 2. A local structure theorem for stacks with self-dual cotangent complex

We will make use of an even more specific local model for stacks of coherent sheaves on a $K3$-surface. We consider an action of an algebraic group $G$ on a smooth affine scheme $\mathrm{Spec}(B)$.

[1] Recall that a geometric stack is a quasi-compact algebraic stack with affine diagonal.

**Definition 2.1.** A *weak co-moment map* is a $G$-equivariant linear map $\mu : \mathfrak{g} \to B$ along with $G$-equivariant isomorphisms $\phi_0 : \Omega^1_B \simeq T_B$ and $\phi_1 : B \otimes \mathfrak{g} \simeq B \otimes \mathfrak{g}$ such that the diagram

$$\begin{tikzcd} B \otimes \mathfrak{g} \arrow[r, "d\mu"] \arrow[d, "\phi_1"] & \Omega^1_B \arrow[d, "\phi_0"] \\ B \otimes \mathfrak{g} \arrow[r, "a"] & T_B \end{tikzcd}$$

commutes after restricting to $(B/B \cdot \mu(\mathfrak{g}))^{red}$, where $a : \mathfrak{g} \to T_B$ is the infinitesimal derivative of the $G$ action on $B$.

**Remark 2.2.** When $\phi_0$ is induced by an algebraic symplectic form, and $\phi_1$ is the identity, the map $\mu$ is called the co-moment map and is uniquely defined up to adding a character of $\mathfrak{g}$. Thus our notion is a weaker version of this more common concept.

Given a weak co-moment map, one can form a commutative DGA $B[\mathfrak{g}[1]; d\xi = \mu(\xi)]$, by which we mean the free graded-commutative $B$-algebra generated by the vector space $\mathfrak{g}$, where the differential on $\mathfrak{g}$ is given by $\mu$. This defines an affine derived scheme which also has a $G$-action.

**Theorem 2.3.** *Let $\mathcal{X}$ be a derived algebraic stack such that $L_{\mathcal{X}} \simeq L_{\mathcal{X}}^\vee$, let $\mathcal{X}^{\mathrm{cl}} \to Y$ be a good moduli space. For any closed point $y \in Y$, there is an étale map*

$$\mathcal{X}' := \mathrm{Spec}(B[\mathfrak{g}[1]; d\xi = \mu(\xi)])/G \to \mathcal{X}$$

*whose image in $Y$ contains $y$, where $G$ is linearly reductive, $\mathrm{Spec}(B)$ is a smooth affine $G$-scheme, and $\mu : \mathfrak{g} \to B$ is a weak co-moment map. Furthermore, if $U = \mathrm{Spec}((B/B \cdot \mu(\mathfrak{g}))^G)$, then $(\mathcal{X}')^{cl} \simeq \mathcal{X}^{cl} \times_Y U$.*

As a basic input to the proof of this theorem, we observe:

**Lemma 2.4.** *Let $\mathcal{X}$ be a derived stack such that $\mathcal{X}^{\mathrm{cl}} \simeq \mathrm{Spec}(R)/G$ for some ring $R$ and reductive group $G$. Then $\mathcal{X} \simeq \mathrm{Spec}(A)/G$ for some connective CDGA $A$ with a $G$-equivariant isomorphism $\pi_0(A) \simeq R$.*

*Proof.* By approximating $\mathcal{X}$ by its truncations, it suffices to show that if $\mathcal{X}' \to \mathcal{X}$ is a square-zero extension and $\mathcal{X}' \simeq \mathrm{Spec}(A')/G$ for some $G$-equivariant CDGA $A'$, then $\mathcal{X} \simeq \mathrm{Spec}(A)/G$ for some square-zero extension of $G$-equivariant CDGA $A \to A'$. This amounts to a deformation theory problem: showing that square-zero extensions of the stack by $\mathrm{Spec}(A')/G$ by a coherent sheaf over $\mathrm{Spec}(\pi_0(A'))/G$ correspond bijectively to $G$-equivariant square zero extensions of $\mathrm{Spec}(A')$ by this same coherent sheaf. This is a consequence of the long exact sequence for the relative cotangent complex of the map $\mathrm{Spec}(A') \to \mathrm{Spec}(A')/G$ and the fact that coherent sheaves on $\mathrm{Spec}(\pi_0(A'))/G$ have vanishing higher cohomology. □

The next useful observation is that Zariski locally over $\mathrm{Spec}(\pi_0(A)^G)$, one can describe certain derived stacks as a *derived complete intersection* [AG]. By this we mean Spec of a $G$-equivariant CDGA of the form $k[U_0, U_1; d]$, which denotes the free CDGA generated by a representation $U_0$ of $G$ in homological degree 0 and a representation $U_1$ in homological degree 1 with a $G$-equivariant differential defined by the $G$-equivariant linear map $d : U_1 \to k[U_0]$.

**Lemma 2.5.** *Let $G$ be a linearly reductive group, and let $A$ be a connective $G$-equivariant CDGA which is quasi-smooth.*[2] *Let $o \in \mathrm{Spec}(\pi_0(A))$ be a closed point with $\mathrm{Aut}(o) = G$. Then there is*

[2]A derived stack is defined to be *quasi-smooth* if the cotangent complex $\mathbb{L}_{\mathcal{X}}$ is a perfect complex with Tor-amplitude in $[1, 0, -1]$.

*an $f \in \pi_0(A)^G$ which does not vanish at $o$ and a $G$-equivariant quasi-isomorphism with a derived complete intersection*

$$A_f \simeq k[U_0, U_1; d],$$

*where $A_f$ is the localization of $A$.*

*Proof.* First choose some $G$-equivariant semi-free presentation $A \sim k[U_\bullet; d_A]$, where $U_\bullet = \bigoplus_{i\geq 0} U_i[i]$ – in other words $U_i$ is placed in homological degree $i$. The fiber of the cotangent complex at $o$ has the form

$$L_A|_o \simeq (\cdots \to U_2 \xrightarrow{d_2} U_1 \xrightarrow{d_1} U_0)$$

Choosing a subspace $W \subset U_1$ which maps isomorphically onto the quotient $U_1/\operatorname{im}(U_2)$, the inclusion of the subcomplex $(W \to U_0) \subset (\cdots \to U_2 \to U_1 \to U_0)$ is a quasi-isomorphism. It follows from the fact that $\operatorname{Spec}(A)$ is quasi-smooth that the inclusion of the semi-free subalgebra $A' := k[U_0, W[1]; d_A] \subset k[U_\bullet; d_A]$ induces an isomorphism of cotangent complexes at $o$. Thus the closed immersion

$$\operatorname{Spec}(A)/G \hookrightarrow \operatorname{Spec}(A')/G$$

is étale at $o \in \operatorname{Spec}(A)$ and hence a Zariski open immersion in a neighborhood of $o$. It follows from the orbit structure of reductive group actions on affine schemes that any $G$-equivariant open neighborhood of a point with a closed orbit is contained in an open affine of the form $A_f$ for some $f \in \pi_0(A)^G$.

□

*Proof of Theorem 2.3.* Using Theorem 1.2 and Lemma 2.4, we may reduce to the case where $\mathcal{X} = \operatorname{Spec}(A)/G$ for a linearly reductive group $G$ and $G$-equivariant connective CDGA $A$. Furthermore we may assume that if $o \in \operatorname{Spec}(\pi_0(A))$ is the unique point with closed orbit in the fiber over $y \in Y$, then $o$ is in fact a closed point and $\operatorname{Aut}(o) = G$. The quasi-isomorphism $L_\mathcal{X} \simeq (L_\mathcal{X})^\vee$ implies that $\mathcal{X}$ is quasi-smooth, so Lemma 2.5 implies that we can further reduce to the case where $A = k[U_0, U_1; d]$ is a $G$-equivariant complete intersection.

The cotangent complex admits an explicit presentation of the form

$$L_{\operatorname{Spec}(A)/G} \simeq A \otimes \left(\delta U_1[1] \oplus \delta U_0 \oplus \mathfrak{g}^\vee[-1]\right)$$

where the vector space $\mathfrak{g}^\vee[-1] \oplus \delta U_0 \oplus \delta U_1[1]$ is concentrated in homological degree $-1, 0, 1$, and the differential is a deformation of the differential internal to $A$. The space $\delta U_i$ is the space of "formal differentials" of elements of $U_i$, and is isomorphic to $U_i$ as a $G$-representation. The hypothesis on $\operatorname{Aut}(o)$ implies that the quasi-isomorphism $\psi : L_\mathcal{X} \simeq (L_\mathcal{X})^\vee$ provides an isomorphism

$$\mathfrak{g} \simeq H_1(L_\mathcal{X}|_o) \subset \delta U_1.$$

We shall consider the resulting splitting $\delta U_1 \simeq U_1 \simeq \mathfrak{g} \oplus W$ as a $G$-representation.

The $G$-equivariant complete intersection CDGA $A' := k[U_0, W; d_A] \subset A$ is smooth at $o \in \operatorname{Spec}(\pi_0(A))$, and hence $A'$ will be smooth and classical after inverting an element $f \in \pi_0(A)^G$. We regard $A$ as obtained from $A'$ by adjoining relations corresponding to a moment map $\mu : \mathfrak{g} \to k[U_0]$, and hence after inverting $f$ we will have a quasi-isomorphism

$$A_f \simeq B[\mathfrak{g}[1]; d\xi = \mu(\xi)],$$

where $B := \pi_0(A'_f)$ is a smooth ring and $\mu : \mathfrak{g} \to k[U_0] \to B$ the induced $G$-equivariant map. This map must be a weak co-moment map by Lemma 2.6 below. □

**Lemma 2.6.** *Let $G$ be a reductive group, let $B$ be a smooth $k$-algebra with a $G$-equivariant map $\mu : \mathfrak{g} \to B$, and let $\mathcal{X} := \operatorname{Spec}(B[\mathfrak{g}[1]; d\xi = \mu(\xi)])/G$. Fix a closed point $o \in \operatorname{Spec}(B/B \cdot \mu(\mathfrak{g}))^G$ at which $\operatorname{Aut}(o) = G$. Then any isomorphism $\psi : L_\mathcal{X} \simeq (L_\mathcal{X})^\vee$ induces, after inverting an element $f \in \pi_0(B)^G$ with $f(o) \neq 0$, isomorphisms $\phi_1 : B \otimes \mathfrak{g} \to B \otimes \mathfrak{g}$ and $\phi_0 : \Omega^1_B \to T_B$ giving $\mu$ the structure of a weak co-moment map.*

*Proof.* The cotangent complex, which is a dg-module over $A := B[\mathfrak{g}[1]; d]$, has the form

$$L_{\mathrm{Spec}(A)/G} = A \otimes \mathfrak{g}[1] \oplus A \otimes_B \Omega^1_B \oplus A \otimes \mathfrak{g}^\vee[-1]$$

where the differential is a deformation of the differential internal to $A$ by the $k$-linear map $d\mu : \mathfrak{g} \to \Omega^1_B$ and the action map $\Omega^1_B \to B \otimes \mathfrak{g}^\vee$ restricted to $A$. Concretely, the isomorphism $\psi : L_{\mathfrak{X}} \simeq (L_{\mathfrak{X}})^\vee$ induces an isomorphism of dg-$A$-modules

$$\begin{tikzcd} L_{\mathrm{Spec}(A)/G} \simeq \arrow[d, "\psi"] & \cdots \arrow[r] & \Omega^1_B \oplus A_1 \otimes \mathfrak{g}^\vee \arrow[r, "a \oplus (d_A \otimes \mathfrak{g}^\vee)"] \arrow[d, "\psi_0"] & A_0 \otimes \mathfrak{g}^\vee \arrow[d, "\psi_{-1}"] . \\ (L_{\mathrm{Spec}(A)/G})^\vee \simeq & \cdots \arrow[r] & T_B \oplus A_1 \otimes \mathfrak{g}^\vee \arrow[r, "(d\mu)^\vee \oplus (d_A \otimes \mathfrak{g}^\vee)"] & A_0 \otimes \mathfrak{g}^\vee \end{tikzcd}$$

Bear in mind that $A_0 = B$, $A_1 = B \otimes \mathfrak{g}$, $A_2 = B \otimes \bigwedge^2 \mathfrak{g}$, etc. Now at the point $o \in \mathrm{Spec}(B/\mu(\mathfrak{g})) \subset \mathrm{Spec}(B)$ the differential in this complex vanishes, so $\psi_{-1}$ induces an isomorphism $B \otimes \mathfrak{g}^\vee \to B \otimes \mathfrak{g}^\vee$ in a neighborhood of $o$ in $\mathrm{Spec}(B)$. Likewise, as $\psi$ is a map of dg-$A$-modules, the map $\psi_0$ maps $A_1 \otimes \mathfrak{g}^\vee$ to $A_1 \otimes \mathfrak{g}^\vee$ and is induced by $\psi_{-1}$. It thus descends to a map $\psi_0 : \Omega^1_B \to T_B$ of $B$-modules. The resulting diagram of maps of $B$-modules

$$\begin{tikzcd} \Omega^1_B \arrow[r, "a"] \arrow[d, "\psi_0"] & B \otimes \mathfrak{g}^\vee \arrow[d, "\psi_{-1}"] \\ T_B \arrow[r, "(d\mu)^\vee"] & B \otimes \mathfrak{g}^\vee \end{tikzcd}$$

commutes after restricting to $B/B \cdot \mu(\mathfrak{g})$, and the vertical arrows are isomorphisms in a $G$-equivariant Zariski-open neighborhood of $o \in \mathrm{Spec}(B)$. After localizing we can invert the vertical arrows and dualize this diagram, giving $\mu$ the structure of a weak co-moment map. $\square$

## 3. The magic windows theorem

We first define our "weight windows." We will assume that for a generic class $\ell \in NS(\mathfrak{X})_{\mathbb{R}}$, the open substack $\mathfrak{X}^{\mathrm{ss}}(\ell) \subset \mathfrak{X}$ provided by Theorem 1.1 is Deligne-Mumford. This will happen in our examples. Let $\mathfrak{X}$ be a derived stack whose cotangent complex is perfect, and let $f : B\mathbb{C}^* \to \mathfrak{X}$ be a map. Then we define

$$a^{\mathfrak{X}}_f := \mathrm{wt}\left(\det((f^* L_{\mathfrak{X}})^{<0})\right)$$

**Definition 3.1.** We say that $\delta \in \mathrm{Pic}(\mathfrak{X})_{\mathbb{R}}$ is *generic* if for all $f : B\mathbb{G}_m \to \mathfrak{X}$, the weight of $f^*\delta - \frac{1}{2} a^{\mathfrak{X}}_f$ is not an integer.

First let us consider the $\Theta$-stratification of Theorem 1.1. We will restrict our attention to maps $f : (B\mathbb{G}_m)_{k'} \to \mathfrak{X}$ classified by $k'$ points of the centers $\mathfrak{Z}^{\mathrm{ss}}_\alpha$ of the strata $\mathfrak{S}_\alpha$, and we call such a map $\ell$-canonical.

**Definition 3.2.** For any $\delta \in NS(\mathfrak{X})_{\mathbb{R}}$, we define the full subcategory

$$\mathcal{G}^\ell_{\mathfrak{X}}(\delta) = \left\{ F \in \mathrm{DCoh}(\mathfrak{X}) \;\middle|\; \begin{array}{c} \text{for all } \ell\text{-canonical maps } f : (B\mathbb{G}_m)_{k'} \to \mathfrak{X} : \\ \mathrm{minWt}(f^*(F)) \geq \mathrm{wt}(f^*\delta) + \frac{1}{2} a^{\mathfrak{X}}_f, \text{ and} \\ \mathrm{minWt}(f^*(\mathbb{D}_{\mathfrak{X}}(F))) \geq \frac{1}{2} a^{\mathfrak{X}}_f - \mathrm{wt}(f^*\delta) \end{array} \right\}$$

**Theorem 3.3.** [HL3, Theorem 3.2] *For $\delta \in NS(\mathfrak{X})_{\mathbb{R}}$ generic and $\ell \in NS(\mathfrak{X})_{\mathbb{R}}$ arbitrary, the restriction functor induces an equivalence $\mathcal{G}^\ell_{\mathfrak{X}}(\delta) \simeq \mathrm{DCoh}(\mathfrak{X}^{\mathrm{ss}}(\ell))$.*

*Proof.* This is a slight rephrasing of the cited theorem. The theorem in [HL3] has an additional hypothesis on the cotangent complex, but it is guaranteed by the isomorphism $L_{\mathcal{X}} \simeq (L_{\mathcal{X}})^{\vee}$. Furthermore, we have used [HL3, Lemma 3.10] to identify our definition of the category $\mathcal{G}^{\ell}_{\mathcal{X}}(\delta)$ with the definition in [HL3]. $\square$

As the notation suggests, the category $\mathcal{G}^{\ell}_{\mathcal{X}}(\delta)$ depends on the class $\ell$, or more specifically the $\Theta$-stratification of Theorem 1.1. But the dependence on $\ell$ only shows up in the definition of $\mathcal{G}^{\ell}_{\mathcal{X}}(\delta)$ where one chooses which maps $f : (B\mathbb{G}_m)_{k'} \to \mathcal{X}$ to apply the weight conditions to.

**Definition 3.4.** Let $\mathcal{X}$ be a derived stack such that $L_{\mathcal{X}} \simeq L_{\mathcal{X}}^{\vee}$. Then for any $\delta \in NS(\mathcal{X})_{\mathbb{R}}$ we define

$$\mathcal{M}_{\mathcal{X}}(\delta) = \left\{ F \in \mathrm{DCoh}(\mathcal{X}) \;\middle|\; \begin{array}{c} \text{for all } f : (B\mathbb{G}_m)_{k'} \to \mathcal{X} \\ \mathrm{minWt}(f^*F) = -\infty \text{ or } \geq \mathrm{wt}(f^*\delta) + \frac{1}{2}a_f^{\mathcal{X}}, \text{ and} \\ \mathrm{minWt}(f^*(\mathbb{D}_{\mathcal{X}}(F))) = -\infty \text{ or } \geq \frac{1}{2}a_f^{\mathcal{X}} - \mathrm{wt}(f^*\delta) \end{array} \right\}$$

**Remark 3.5.** For a general map $f : (B\mathbb{G}_m)_{k'} \to \mathcal{X}$ and $F \in \mathrm{DCoh}(\mathcal{X})$, the weights of $f^*(F) \in \mathrm{D}^-\mathrm{Coh}((B\mathbb{G}_m)_{k'})$ need not be bounded below. However, one consequence of [HL3, Theorem 3.2] is that for any map $f : (B\mathbb{G}_m)_{k'} \to \mathcal{X}$ classified by a $k'$-point of $\mathcal{Z}^{\mathrm{ss}}_{\alpha}$ for some $\alpha$ and any $F \in \mathrm{DCoh}(\mathcal{X})$, the weights of $f^*(F)$ are bounded below even though the complex need not be homologically bounded. It follows that $\mathcal{M}_{\mathcal{X}}(\delta) \subset \mathcal{G}^{\ell}_{\mathcal{X}}(\delta)$ for any $\ell$ and $\delta$.

**Theorem 3.6** (Magic windows)**.** *Let $\mathcal{X}$ be a derived stack such that $L_{\mathcal{X}} \simeq L_{\mathcal{X}}^{\vee}$ and $\mathcal{X}^{\mathrm{cl}}$ admits a good moduli space. Let $\delta \in NS(\mathcal{X})_{\mathbb{R}}$ be generic, and assume that $\ell \in NS(\mathcal{X})_{\mathbb{R}}$ is such that $\mathcal{X}^{\mathrm{ss}}(\ell)$ is Deligne-Mumford. Then*

$$\mathcal{M}_{\mathcal{X}}(\delta) = \mathcal{G}^{\ell}_{\mathcal{X}}(\delta)$$

*as subcategories of* $\mathrm{DCoh}(\mathcal{X})$*, and hence the restriction functor defines an equivalence* $\mathcal{M}_{\mathcal{X}}(\delta) \simeq \mathrm{DCoh}(\mathcal{X}^{\mathrm{ss}}(\ell))$.

**Remark 3.7.** If $\mathcal{X}^{\mathrm{ss}}(\ell)$ is Deligne-Mumford, then the self duality $L_{\mathcal{X}} \simeq (L_{\mathcal{X}})^{\vee}$ implies that $\mathcal{X}^{\mathrm{ss}}(\ell)$ is in fact smooth.

**Remark 3.8.** The condition that there exists a $\delta \in NS(\mathcal{X})_{\mathbb{R}}$ which is generic in the sense above is non-vacuous. One can show that $\delta \in NS(\mathcal{X})_{\mathbb{R}}$ is generic if and only if for every closed point $x \in \mathcal{X}$ with linearly reductive automorphism group $G$, the restriction of $\delta$ to a real character of a maximal torus $T \subset G$ is generic in the sense of [HLS]. The results of [HLS], combined with the local description of $\mathcal{X}$ above, can be used to show that if there exists a generic $\delta$ then $\mathcal{X}^{\mathrm{ss}}(\ell)$ is Deligne-Mumford for $\ell$ in the complement of a finite real hyperplane arrangement in $NS(\mathcal{X})_{\mathbb{R}}$.

The key value of this theorem is that the category $\mathcal{M}_{\mathcal{X}}(\delta)$ does not depend on $\ell$, unlike the category of $\mathcal{G}^{\ell}_{\mathcal{X}}(\delta)$. An immediate consequence of Theorem 3.6 combined with Theorem 3.3 is the following

**Corollary 3.8.1.** $\mathrm{DCoh}(\mathcal{X}^{\mathrm{ss}}(\ell)) \simeq \mathrm{DCoh}(\mathcal{X}^{\mathrm{ss}}(\ell'))$ *for any two generic $\ell, \ell' \in NS(\mathcal{X})_{\mathbb{R}}$.*

*Proof of Theorem 3.6.* The claim of the theorem is that if $F \in \mathrm{DCoh}(\mathcal{X})$ is a complex for which $f^*(F)$ satisfies certain weight bounds for a special class of maps $f : (B\mathbb{G}_m)_{k'} \to \mathcal{X}$ – namely those maps classified by a point of $\mathcal{Z}^{\mathrm{ss}}_{\alpha}(k')$ for some $\alpha$ – then $f^*(F)$ satisfies the analogous weight bounds (or has unbounded weights) with respect to *any* map $f : (B\mathbb{G}_m)_{k'} \to \mathcal{X}$.

Both of these weight conditions on the complex $F$ can be checked étale locally over the good moduli space of $\mathcal{X}^{\mathrm{cl}}$, and Theorem 1.1 implies that the formation of the stratification of $\mathcal{X}$ is étale local over the good moduli space of $\mathcal{X}^{\mathrm{cl}}$ as well. Thus it suffices to prove the claim after base change along an étale cover of the good moduli space. Theorem 2.3 implies that we may assume that $\mathcal{X} = \mathrm{Spec}(B[\mathfrak{g}[1]; d\xi = \mu(\xi)])$ for some linearly reductive $G$, smooth equivariant $G$-algebra $B$, and weak co-moment map $\mu : \mathfrak{g} \to B$. This is the situation in which we prove the theorem, in the next section. $\square$

3.1. **Proof of the magic windows theorem in the local case.** This is the technical heart of our main result. We work with a smooth $G$-scheme $X := \operatorname{Spec}(R)$ along with a weak co-moment map $\mu : \mathfrak{g} \to R$, which we regard as a map $X \to \mathfrak{g}^\vee$. From this data we construct the scheme $Y = X \times \mathfrak{g}$ along with the map $W : Y \to \mathbb{A}^1$ given by $W(x, \xi) = \langle \mu(x), \xi \rangle$. $W$ is equivariant with respect to scaling $\mathfrak{g}$ and $\mathbb{A}^1$ with weight 1.

Our proof of Theorem 3.6 in the local case will make use of the "Landau-Ginzburg/Calabi-Yau" correspondence, which is a correspondence between the two geometric objects:

(1) the "Hamiltonian reduction" stack $\mathcal{X}_0 := X_0/G$, where $X_0 := \mu^{-1}(0) = \operatorname{Spec}(R[\mathfrak{g}[1]; d\xi = \mu(\xi)])$ is the derived zero fiber of the weak moment map ($\mathcal{X}_0$ is what is referred to as $\mathcal{X}$ in the other sections of this paper), and
(2) the "graded LG model" $(\mathcal{Y}, W)$ with $\mathcal{Y} := Y/G \times \mathbb{G}_m$, where $\mathbb{G}_m$ acts on $Y$ by scaling $\mathfrak{g}$ with weight 1, and the map $W : \mathcal{Y} \to \mathbb{A}^1/\mathbb{G}_m$ above.

We will also denote the derived zero fiber $Y_0 := W^{-1}(0) = \operatorname{Spec}(R[\mathfrak{g}^\vee, \epsilon[1]; d\epsilon = W])$ and consider the stack $\mathcal{Y}_0 := Y_0/G \times \mathbb{G}_m$.

We will continue to study the derived category of complexes with bounded coherent cohomology over $\mathcal{X}_0$,

$$\operatorname{DCoh}(\mathcal{X}_0) \simeq \operatorname{DCoh}^G(X_0).$$

On the other hand, we can consider the "graded singularity category"

$$\operatorname{DSing}(\mathcal{Y}, W) \simeq \operatorname{DCoh}(\mathcal{Y}_0)/\operatorname{Perf}(\mathcal{Y}_0) \simeq \operatorname{DCoh}^{G\times\mathbb{G}_m}(Y_0)/\operatorname{Perf}^{G\times\mathbb{G}_m}(Y_0).$$

The superscript $G$ or $G \times \mathbb{G}_m$ indicates that we are working with categories of equivariant complexes. See [HLP] for a discussion of the graded singularity category.

**Theorem 3.9.** [S, I] *There is a canonical equivalence* $\Phi_X : \operatorname{DSing}^{G\times\mathbb{G}_m}(Y, W) \simeq \operatorname{DCoh}^G(X_0)$.

We will prove our theorem using this equivalence to convert the question on $\operatorname{DCoh}^G(X_0)$ to a question about $\operatorname{DSing}^{G\times\mathbb{G}_m}(Y, W)$. We will introduce subcategories $\mathcal{M}_{(\mathcal{Y},W)}(\delta) \subset \mathcal{G}^\ell_{(\mathcal{Y},W)}(\delta) \subset \operatorname{DSing}(\mathcal{Y}, W)$, defined precisely in Definition 3.11 and Definition 3.15 below, which correspond to the categories $\mathcal{M}_{\mathcal{X}_0}(\delta)$ and $\mathcal{G}^\ell_{\mathcal{X}_0}(\delta)$ under $\Phi_X$. The strategy of the proof can be described by the following commutative diagram, in which *res* denotes a restriction functor:

$$\begin{tikzcd}
\mathcal{M}_{(\mathcal{Y},w)}(\delta) \arrow[d, dashed, "④"] & \overset{③}{\subset} & \mathcal{G}^\ell_{(\mathcal{Y},W)}(\delta) \arrow[d, dashed, "④"] \arrow[rrr, bend left, "①"] \arrow[ddrrrr, bend right, "\simeq \text{ by } \textit{Theorem 3.3}"'] & \subset & \operatorname{DSing}(\mathcal{Y}, W) \arrow[r, "res"] \arrow[d, "\Phi_X", "\simeq"'] & \operatorname{DSing}(\mathcal{Y}^{ss}, W) \arrow[r, "res", "②"'] & \operatorname{DSing}^G(X^{ss} \times \mathfrak{g}, W) \arrow[d, "\Phi_{X^{ss}}", "\simeq"'] \\
\mathcal{M}_{\mathcal{X}_0}(\delta) & \subset & \mathcal{G}^\ell_{\mathcal{X}_0}(\delta) & \subset & \operatorname{DCoh}(\mathcal{X}_0) \arrow[rr, "res"] & & \operatorname{DCoh}(\mathcal{X}_0^{ss})
\end{tikzcd} \quad (1)$$

The goal is to show that $\mathcal{M}_{\mathcal{X}_0}(\delta) \subset \mathcal{G}^\ell_{\mathcal{X}_0}(\delta)$ is an equality, assuming $\delta$ is generic, which will follow automatically after verifying the following claims (referencing the diagram above):

(1) The restriction map (1) is an equivalence,
(2) The restriction map (2) is an equivalence,
(3) The inclusion (3) is an equality, and
(4) The equivalence $\Phi_X$ maps the subcategories of $\operatorname{DSing}^{G\times\mathbb{G}_m}(Y, W)$ to the corresponding subcategories of $\operatorname{DCoh}^G(X_0)$.

The most subtle point, which requires us to analyze the equivalence $\Phi_X$ carefully, will be (4).

**Remark 3.10.** One of the stated assumptions of **??** is that $X_0^{\mathrm{ss}}(\ell)/G$ is Deligne-Mumford. We can pull back $\ell \in NS(X/G)_{\mathbb{R}}$ to $Y/G = X \times \mathfrak{g}/G$, and the $\ell$-semistable locus $Y^{ss}$ contains $X^{\mathrm{ss}} \times \mathfrak{g}$. The existence of a generic parameter $\delta$ implies that we may perturb $\ell$ slightly so that $(X \times \mathfrak{g})^{\mathrm{ss}}(\ell)/G$ will be Deligne-Mumford without changing $X_0^{\mathrm{ss}}(\ell)$ or the GIT stratification of $X_0$, so we will also assume without loss of generality that $(X \times \mathfrak{g})^{\mathrm{ss}}(\ell)/G$ is Deligne-Mumford as well.

3.1.1. *The proof of* (1). First we recall that $\mathrm{DSing}^{G\times\mathbb{G}_m}(Y, W)$ is obtained from $\mathrm{DCoh}^{G\times\mathbb{G}_m}(Y_0)$ by a localization procedure which inverts a certain natural transformation $\beta : F \otimes \mathcal{L}[-2] \to F$ for $F \in \mathrm{DCoh}^{G\times\mathbb{G}_m}(Y_0)$. The result is a dg-enhancement of the Verdier quotient $\mathrm{DCoh}(\mathcal{Y}_0)/\operatorname{Perf}(\mathcal{Y}_0)$. What's relevant for our purposes is that $\mathrm{DSing}^{G\times\mathbb{G}_m}(Y, W)$ is generated by objects from $\mathrm{DCoh}^{G\times\mathbb{G}_m}(Y_0)$.

For any $f : (B\mathbb{G}_m)_{k'} \to \mathcal{Y}$ we define the integer

$$\eta_f^{\mathcal{Y}} := \mathrm{wt}\left(\det((f^* L_{\mathcal{Y}})^{>0})\right)$$

We note that $\ell$ induces a $\Theta$-stratification $\mathcal{Y} = \mathcal{Y}^{\mathrm{ss}} \cup \mathcal{S}_0^{\mathcal{Y}} \cup \cdots \cup \mathcal{S}_N^{\mathcal{Y}}$ as in Theorem 1.1. We will call a map $f : (B\mathbb{G}_m)_{k'} \to \mathcal{Y}$ *$\ell$-canonical* if it is classified by a point in the center of one of the strata $\mathcal{S}_\alpha^{\mathcal{Y}}$. The closed substack $\mathcal{Y}_0 \subset \mathcal{Y}$ inherits a $\Theta$-stratification as well.

**Definition 3.11.** We define the full subcategory $\mathcal{G}_{\mathcal{Y}_0}^\ell(\delta) \subset \mathrm{DCoh}(\mathcal{Y}_0)$,

$$\mathcal{G}_{\mathcal{Y}_0}^\ell(\delta) = \left\{ F \in \mathrm{DCoh}(\mathcal{Y}_0) \;\middle|\; \begin{array}{c} \text{for all } \ell\text{-canonical } f : (B\mathbb{G}_m)_{k'} \to Y_0/G \\ \mathrm{minWt}(f^*(F)) \geq \mathrm{wt}(f^*\delta) - \frac{1}{2}\eta_f^{\mathcal{Y}}, \text{ and} \\ \mathrm{minWt}(f^*(\mathbb{D}_{\mathcal{X}}(F))) \geq -\frac{1}{2}\eta_f^{\mathcal{Y}} - \mathrm{wt}(f^*\delta) \end{array} \right\}.$$

We define $\mathcal{G}_{(\mathcal{Y},W)}^\ell(\delta) \subset \mathrm{DSing}(\mathcal{Y}, W)$ to be the full subcategory generated by the image of $\mathcal{G}_{\mathcal{Y}_0}^\ell(\delta)$ under the quotient functor $\mathrm{DCoh}(\mathcal{Y}_0) \to \mathrm{DSing}(\mathcal{Y}, W)$.

**Remark 3.12.** Because $Y_0 \hookrightarrow Y$ is a hypersurface in a smooth variety defined by a function which is $G$-invariant, we will see below that the condition that $F \in \mathrm{DCoh}(\mathcal{Y}_0)$ lies in $\mathcal{G}_{\mathcal{Y}_0}^\ell(\delta)$ is equivalent to requiring that for all $\ell$-canonical maps $f : (B\mathbb{G}_m)_{k'} \to Y_0/G$ the weights of $f^*(F)$ lie in the interval

$$\mathrm{wt}(f^*\delta) + \frac{1}{2}[-\eta_f^{\mathcal{Y}}, \eta_f^{\mathcal{Y}}]$$

**Proposition 3.13.** [HL2, Section 4] *The restriction functor* $\mathrm{DCoh}(\mathcal{Y}_0) \to \mathrm{DCoh}(\mathcal{Y}_0^{\mathrm{ss}})$ *induces equivalences*

$$\mathcal{G}_{\mathcal{Y}_0}^\ell \simeq \mathrm{DCoh}(\mathcal{Y}_0^{\mathrm{ss}}), \; \textit{and}$$
$$\mathcal{G}_{\mathcal{Y}_0}^\ell \cap \operatorname{Perf}(\mathcal{Y}_0) \simeq \operatorname{Perf}(\mathcal{Y}_0^{\mathrm{ss}})$$

The key reason this holds for both coherent and perfect complexes is that $W$ has weight 0 with respect to all $\ell$-canonical maps $(B\mathbb{G}_m)_{k'} \to \mathcal{Y}$.

3.1.2. *The proof of* (2). The subtlety is that the inclusion $X^{\mathrm{ss}} \times \mathfrak{g} \subset (X \times \mathfrak{g})^{\mathrm{ss}}$ is not an equality. For graded singularity categories, though, restriction to an open substack induces an equivalence of categories whenever the open substack contains the entire critical locus of $W$. It therefore suffices to verify

**Lemma 3.14.** *If $\mu : X \to \mathfrak{g}^\vee$ is a weak co-moment map, inducing a function $W : X \times \mathfrak{g} \to \mathbb{A}^1$, then*

$$\mathrm{Crit}(W)|_{(X\times\mathfrak{g})^{\mathrm{ss}}} \subset X^{\mathrm{ss}} \times \mathfrak{g}$$

*Proof.* From the defining formula $W(x,\xi) = \langle \mu(x), \xi \rangle$, we see that $dW_{(x,\xi)} = \langle d\mu(x), \xi \rangle \oplus \mu(x) \in \Omega^1_{X,x} \oplus \mathfrak{g}^\vee$. Note that for a weak co-moment map, at any point of $x$ the linear map $d\mu : T_{X,x} \to \mathfrak{g}^\vee$ is isomorphic to the action map $a : \Omega^1_{X,x} \to \mathfrak{g}^\vee$, so $\langle d\mu(x), \xi \rangle = 0$ if and only if $\xi \in \mathrm{Lie}(\mathrm{Stab}(x))$. It follows that

$$dW_{(x,\xi)} = 0 \quad \Leftrightarrow \quad \mu(x) = 0 \text{ and } \xi \in \mathrm{Lie}(\mathrm{Stab}(x)).$$

If such a point $(x,\xi) \in X \times \mathfrak{g}$ is $\ell$-semistable, then $x$ must be semistable as well. If not, then one could consider the canonical maximal destabilizing subgroup $\lambda : \mathbb{G}_m \to G$ for $x$, and because $\xi \in \mathrm{Lie}(\mathrm{Stab}(x))$ the limit $\lim_{t\to 0} \lambda(t)\cdot(x,\xi)$ would exist. Thus $\lambda$ would be destabilizing for $(x,\lambda)$ according to the Hilbert-Mumford criterion. $\square$

3.1.3. *The proof of* (3). We first must formulate the magic window subcategory of $\mathrm{DSing}(\mathcal{Y}, W)$. As before, the only difference between the magic window category $\mathcal{M}$ and the category $\mathcal{G}$ lies in the choice of which maps $f : (B\mathbb{G}_m)_{k'} \to Y_0/G$ one uses to test the weight conditions.

**Definition 3.15.** We define

$$\mathcal{M}_{\mathcal{Y}_0}(\delta) = \left\{ F \in \mathrm{DCoh}(\mathcal{Y}_0) \;\middle|\; \begin{array}{c} \text{for all maps } f : (B\mathbb{G}_m)_{k'} \to Y_0/G \\ \mathrm{minWt}(f^*(F)) \geq \mathrm{wt}(f^*\delta) - \frac{1}{2}\eta_f^{\mathcal{Y}}, \text{ and} \\ \mathrm{minWt}(f^*(\mathbb{D}_{\mathcal{X}}(F))) \geq -\frac{1}{2}\eta_f^{\mathcal{Y}} - \mathrm{wt}(f^*\delta) \end{array} \right\},$$

and let $\mathcal{M}_{(\mathcal{Y},W)}(\delta) \subset \mathrm{DSing}(\mathcal{Y}, W)$ denote the category generated by the image of $\mathcal{M}_{\mathcal{Y}_0}(\delta)$.

**Remark 3.16.** As remarked above, the weight condition on $F$ can be rephrased by requiring that the weights of $f^*(F)$ lie in the interval $\mathrm{wt}(f^*\delta) + \frac{1}{2}[-\eta_f^{\mathcal{Y}}, \eta_f^{\mathcal{Y}}]$.

**Theorem 3.17.** *If $\delta$ is generic, then $\mathcal{M}_{(\mathcal{Y},W)}(\delta) = \mathcal{G}^{\ell}_{(\mathcal{Y},W)}(\delta)$.*

*Proof.* It suffices to show that $\mathcal{M}_{\mathcal{Y}_0}(\delta) = \mathcal{G}^{\ell}_{\mathcal{Y}_0}(\delta)$ as subcategories of $\mathrm{DCoh}(\mathcal{Y}_0)$, because these categories generate the corresponding subcategories of $\mathrm{DSing}(\mathcal{Y}, W)$ by definition. We shall make use of the magic windows theorem on the smooth stack $\mathcal{Y} = Y/G \times \mathbb{G}_m$, which says that we have an equality $\mathcal{M}_{\mathcal{Y}}(\delta) = \mathcal{G}^{\ell}_{\mathcal{Y}}(\delta)$ as subcategories of $\mathrm{Perf}(\mathcal{Y})$, where the subcategories $\mathcal{M}_{\mathcal{Y}}(\delta) \subset \mathcal{G}^{\ell}_{\mathcal{Y}}(\delta) \subset \mathrm{Perf}(\mathcal{Y})$ are defined to consist of complexes satisfying the same weight bounds with respect to maps $(B\mathbb{G}_m)_{k'} \to \mathcal{Y}$ as in the definition of $\mathcal{M}_{\mathcal{Y}_0}(\delta) \subset \mathcal{G}^{\ell}_{\mathcal{Y}_0}(\delta)$ above. The equality $\mathcal{M}_{\mathcal{Y}}(\delta) = \mathcal{G}^{\ell}_{\mathcal{Y}}(\delta)$ was proved in [HLS] under the more restrictive hypothesis that $X = \mathbb{A}^n$ is a linear representation of $G$, but one can reduce the general case of a smooth affine quotient stack to this one via Luna's étale slice theorem.

The category $\mathrm{Perf}(\mathcal{Y}_0^{\mathrm{ss}})$ is generated by the restriction of perfect complexes on $\mathcal{Y}^{\mathrm{ss}} = (X \times \mathfrak{g})^{\mathrm{ss}}/G \times \mathbb{G}_m$. Combining this with Proposition 3.13 and the magic windows theorem, which implies that $\mathcal{M}_{\mathcal{Y}}(\delta) \to \mathrm{DCoh}(\mathcal{Y}^{\mathrm{ss}})$ is an equivalence, we see that the image of the functor

$$\mathcal{M}_{\mathcal{Y}}(\delta) \to \mathrm{Perf}(\mathcal{Y}_0) \cap \mathcal{G}^{\ell}_{\mathcal{Y}_0}(\delta)$$

generates the latter category. This shows that

$$\mathrm{Perf}(\mathcal{Y}_0) \cap \mathcal{G}^{\ell}_{\mathcal{Y}_0}(\delta) = \mathrm{Perf}(\mathcal{Y}_0) \cap \mathcal{M}_{\mathcal{Y}_0}(\delta).$$

In order to conclude that $\mathcal{G}^{\ell}_{\mathcal{Y}_0}(\delta) = \mathcal{M}_{\mathcal{Y}_0}(\delta)$, we use the fact that the functor of Proposition 3.13 which lifts a coherent complex $F \in \mathrm{DCoh}(\mathcal{Y}_0^{\mathrm{ss}})$ to a complex $\tilde{F} \in \mathcal{G}_{\mathcal{Y}_0}(\delta)$ is uniformly bounded in cohomological amplitude, i.e. there is some $K$ such that

$$\underline{H}^i(F) = 0 \text{ for } i > n \Rightarrow \underline{H}^i(\tilde{F}) = 0 \text{ for } i > n + K.$$

Using this and the fact that any complex in $\mathrm{DCoh}(\mathcal{Y}_0^{\mathrm{ss}})$ can be approximated to arbitrarily low cohomological degree by a perfect complex implies that any complex $F \in \mathcal{G}^{\ell}_{\mathcal{Y}_0}(\delta)$ can be approximated to arbitrarily low cohomological degree by a complex in $F' \in \mathrm{Perf}(\mathcal{Y}_0) \cap \mathcal{G}^{\ell}_{\mathcal{Y}_0}(\delta)$. We have already shown that $F' \in \mathcal{M}_{\mathcal{Y}_0}(\delta)$, and for any $f : (B\mathbb{G}_m)_{k'} \to Y_0/G$ the weights of $f^*(F)$ and $f^*(F')$ agree in arbitrarily low cohomological degree. We conclude that $F \in \mathcal{M}_{\mathcal{Y}_0}(\delta)$ as well. $\square$

3.1.4. *The proof of* (4) *and linear Koszul duality.* Let $p : \mathcal{X}_0 \hookrightarrow \mathcal{X} = X/G$ denote the closed immersion, which is a regular embedding. Because $\mathcal{X}$ is smooth, $p_*(F)$ is a perfect complex even when $F$ is not. We will start by considering the weights of complexes $p_*(\Phi_X(F))$ for $F \in \mathcal{M}_{(\mathcal{Y},W)}(\delta)$ by studying the equivalence $\Phi_X : \mathrm{DSing}(\mathcal{Y}, W) \simeq \mathrm{DCoh}(\mathcal{X}_0)$ in more detail.

$\Phi_X$ is constructed using *linear Koszul duality.* Regarding $W$ as an element of $R \otimes \mathfrak{g}^\vee \subset R[\mathfrak{g}^\vee]$, we consider the $G$-equivariant graded CDGA over $R$,

$$\mathcal{B} = R[\mathfrak{g}^\vee, \epsilon[1]; d\epsilon = -W]$$

where $\epsilon$ has homological degree 1, $\mathfrak{g}^\vee$ has homological degree 0, and both have internal degree 1 with respect to the auxiliary $\mathbb{G}_m$ action defining the grading. We then consider the Koszul dual CDGA defined in [I]

$$\mathcal{A} = R[\mathfrak{g}[-1], \beta[-2]; d\xi = \mu(\xi) \cdot \beta, d\beta = 0].$$

where $\mathfrak{g}$ is in cohomological degree 1 and $\beta$ in cohomological degree 2, and both are in internal degree $-1$. The actual statement of the equivalence is the following

**Theorem 3.18.** [I] *There is an $R$-linear $G$-equivariant equivalence*

$$\Phi_X : \mathrm{DCoh}^{G\times\mathbb{G}_m}(\mathcal{B}) \simeq \mathrm{DCoh}^{G\times\mathbb{G}_m}(\mathcal{A})$$

*which identifies subcategory of perfect dg-$\mathcal{B}$-modules with the full subcategory of dg-$\mathcal{A}$-modules which are $\beta$-torsion (i.e. annihilated by $\beta^n$ for some $n$).*

Isik also identifies the Verdier quotient of the category $\mathrm{DCoh}^{G\times\mathbb{G}_m}(\mathcal{A})$ by the category of $\beta$-torsion complexes with the category $\mathrm{DCoh}(\mathcal{X}_0)$ – technically one must mix the internal grading with the homological grading of a $\mathcal{A}[\beta^{-1}]$-module in order to identify this category with the category of modules over the CDGA $R[\mathfrak{g}[1]; d\xi = \mu(\xi)]$. Hence the equivalence $\Phi_X$ descends to an equivalence

$$\mathrm{DSing}(\mathcal{Y}, W) = \mathrm{DCoh}^{G\times\mathbb{G}_m}(\mathcal{B}) / \mathrm{Perf}^{G\times\mathbb{G}_m}(\mathcal{B}) \xrightarrow{\simeq} \mathrm{DCoh}(\mathcal{X}_0)$$

as claimed above. The key property of $\Phi_X$ we use is that its formation is local over $X$ in the sense that if we restrict $\mathcal{A}$ and $\mathcal{B}$ along a map $S \to X$, then the restriction of modules to $S$ intertwines the corresponding equivalences $\Phi_X$ and $\Phi_S$. We now arrive at the main technical result of this paper:

**Proposition 3.19.** *If $F \in \mathcal{M}_{(\mathcal{Y},W)}(\delta)$, then $\Phi_X(F) \in \mathrm{DCoh}(\mathcal{X}_0)$ lies in $\mathcal{M}_{\mathcal{X}_0}(\delta)$.*

We will prove this after establishing a few useful lemmas, working in the more concrete setting of equivariant derived categories. Up to isomorphism every map $f : (B\mathbb{G}_m)_{k'} \to Y_0/G$ corresponds to a pair $(y, \lambda)$, where $\lambda : (\mathbb{G}_m)_{k'} \to G_{k'}$ is a group homomorphism and $y$ is a $k'$-point of $Y_0$ fixed by $\lambda$. Analogously a map $(B\mathbb{G}_m)_{k'} \to \mathcal{X}$ classifies a pair $(x, \lambda)$ with $x \in X$ fixed by $\lambda$.

**Lemma 3.20.** *Let $x \in X$ be such that $\mu(x) = 0$, let $y = (x, 0) \in Y_0$, and let $F \in \mathrm{DCoh}^{G\times\mathbb{G}_m}(Y_0)$ be such that $F|_y$ has $\lambda$-weights in the window*

$$\mathrm{wt}(f^*\delta) + \frac{1}{2}[-\eta_f^{\mathcal{Y}}, \eta_f^{\mathcal{Y}}].$$

*Then $p_*(\Phi_X(F))|_x$ has $\lambda$-weights in the same window.*

*Proof.* Note that the functor $p_*$ simply takes an equivariant $R[\mathfrak{g}[1]; d]$-module and regards it as an equivariant $R$-module. The compatibility of $\Phi_X$ with change of base implies that we may restrict to the fiber over the point $x \in X$. In this case, because $\mu(x) = 0$ be base change of our algebras have trivial differentials, i.e. they are just the free graded CDGA's

$$\mathcal{A}_x = k'[\mathfrak{g}[-1], \beta[-2]] \quad \text{and} \quad \mathcal{B}_x = k'[\mathfrak{g}^\vee, \epsilon[1]]$$

where the homological degree and internal degree of the generators is as above. In addition to the internal and homological grading, the original algebras were $G$-equivariant. $G$ does not necessarily fix the point $x \in X$, but $(\mathbb{G}_m)_{k'}$ acts on $\mathcal{A}_x$ and $\mathcal{B}_x$ via the homomorphism $\lambda$.

For the moment we can disregard the original geometric set up from before and consider the stack $\{x\}/\mathbb{G}_m$ and the $\mathbb{G}_m$-equivariant map $\mu : \{x\} \to \mathfrak{g}^\vee$, where $\mathbb{G}_m$ acts on $\mathfrak{g}^\vee$ via $\lambda$ and the coadjoint representation. This map happens to be 0, but we can still consider the derived fiber of the zero map $W = 0 : \{x\} \times \mathfrak{g} \to \mathbb{A}^1$, which is precisely the CDGA $\mathcal{B}_x$. Likewise $\mathcal{A}_x$ is the Koszul dual CDGA, which after inverting $\beta$ corresponds to the derived zero fiber of $\mu = 0$.

The restriction from $\mathcal{A}$-modules and $\mathcal{B}$-modules to $\mathcal{A}_x$ and $\mathcal{B}_x$ modules intertwines $\Phi_X$ with the usual Koszul duality isomorphism

$$\Phi_x : \mathrm{DCoh}^{G\times\mathbb{G}_m}(k'[\mathfrak{g}^\vee, \epsilon[1]]) \simeq \mathrm{DCoh}^{G\times\mathbb{G}_m}(k'[\mathfrak{g}[-1], \beta[-2]]),$$

where $G = \mathbb{G}_m$ here, but we are distinguishing it notationally from the auxiliary $\mathbb{G}_m$-equivariance corresponding to the internal grading on $\mathcal{A}_x$ and $\mathcal{B}_x$. It suffices to prove the claim for $\Phi_x$, because $F|_{\{x\}\times\mathfrak{g}}$ will still be a coherent complex in $\mathrm{DCoh}^{G\times\mathbb{G}_m}(\mathcal{B}_x)$ whose restriction to $(x, 0) = \{x\} \times \{0\}$ has $G$-weights in the given window.

Any $F \in \mathrm{DCoh}^{G\times\mathbb{G}_m}(\mathcal{B}_x)$ has a minimal resolution of the form $F \sim (\bigoplus_{i\leq k} \mathcal{B}_x \otimes M^i, d)$ where $M^i$ is a $G \times \mathbb{G}_m$-representation in cohomological degree $i$ with the property that the differential $d$ is 0 after restricting to $(x, 0)$. It follows that the weights appearing in $M^\bullet$ are precisely those appearing in $F|_{(x,0)}$, and in particular the $G$-weights of $M^i$ all lie in the given window. We must show that the composition

$$\Psi : \mathrm{DCoh}^{G\times\mathbb{G}_m}(\mathcal{B}_x) \xrightarrow{\Phi_x} \mathrm{DCoh}^{G\times\mathbb{G}_m}(\mathcal{A}_x) \to \mathrm{DCoh}^{G\times\mathbb{G}_m}(\mathcal{A}_x[\beta^{-1}]) \simeq \mathrm{DCoh}^G(k'[\mathfrak{g}[1]]) \qquad (2)$$ {{eqn:koszul

has $G$-weights lying in that same window after forgetting the $k'[\mathfrak{g}[1]]$-module structure (forgetting the $k'[\mathfrak{g}[1]]$-module structure corresponds to pushing forward along $p : \mathcal{X}_0 \to \mathcal{X}$).

Note that the $i^{th}$ term of this complex is $k'[\mathfrak{g}^\vee] \otimes M^i \oplus k'[\mathfrak{g}^\vee]\epsilon \otimes M^{i+1}$. The resolution will eventually be two-periodic in the sense that $M^i \simeq M^{i+2}\langle 1\rangle$ for $i \ll 0$, where $\langle 1\rangle$ denotes the shift by $-1$ in internal degree, and under this identification the differential $d : k'[\mathfrak{g}^\vee] \otimes M^i \to k'[\mathfrak{g}^\vee]\epsilon M^{i+2}$ is multiplication by $\epsilon$. By Theorem 3.18, we know that any perfect complex is annihilated by the composition of functors (2), so we may perform a naive truncation by simply discarding all $M^i$ for $i > 2p$ for some $p << 0$. Thus we can replace $F$ by a complex $F'$ which is actually two-periodic

$$F' \simeq (\bigoplus_{k\geq p} \mathcal{B}_x \otimes (M^{even}[2k]\langle -k\rangle \oplus M^{odd}[2k+1]\langle -k\rangle), d)$$

and such that $\Psi(F) \simeq \Psi(F')$ and the differential $d$ is simply multiplication by $\epsilon$. Let $M^{even} = \bigoplus_n M^{even}_n$ and $M^{odd} = \bigoplus_n M^{odd}_n$ denote the decomposition of $M^{even}$ and $M^{odd}$ into weight spaces as a $G$-representation. Then $\Psi(F') \in \mathrm{DCoh}^G(k[\mathfrak{g}[1]])$ will be quasi-isomorphic to a direct sum of modules of the form $M^{even}_n$ and $M^{odd}_n$, where $\mathfrak{g}[1]$ acts trivially, and the homological shift is determined by mixing the homological and internal grading of the original complex. The relevant thing is that forgetting the $k'[\mathfrak{g}[1]]$-module structure we see that the resulting complex has only those $G$-weights which originally appear in $F|_{(x,0)}$, and this gives the claim. □

**Corollary 3.20.1.** *For any $F \in \mathcal{M}_{(\mathcal{Y},W)}(\delta)$ the complex $p_*(\Phi_X(F)) \in \mathrm{DCoh}(\mathcal{X})$ has the property that for any map $f : (B\mathbb{G}_m)_{k'} \to \mathcal{X}$, the weights of $f^*(p_*(\Phi_X(F)))$ lie in the interval*

$$\mathrm{wt}(f^*\delta) + \frac{1}{2}[-\eta^{\mathcal{Y}}_f, \eta^{\mathcal{Y}}_f]$$

*Proof.* If $f$ does not land in $\mathcal{X}_0$, then $f^*(p_*(\Phi_X(F))) \simeq 0$ and there is nothing to check. If $f$ factors through $\mathcal{X}_0$ then it corresponds to a one parameter subgroup $\lambda : (\mathbb{G}_m)_{k'} \to G_{k'}$ and a point $x \in X(k')$ fixed by $\lambda$ with $\mu(x) = 0$, and we apply the previous lemma. □

Next we introduce a lemma which allows one to compare the weights of $\Phi_X(F)$ with the weights of $p_*(\Phi_X(F))$. We fix a map $f : (B\mathbb{G}_m)_{k'} \to \mathcal{X}_0$ corresponding to a pair $(x, \lambda)$ with $x \in X_0$ as above.

**Lemma 3.21.** *For any $E \in \mathrm{DCoh}^G(X_0)$ we have*

$$\mathrm{minWt}(E_x) = \mathrm{minWt}(p_*(E)_x) - \mathrm{wt}(\det(\mathfrak{g}^{<0})) \text{ or } = -\infty,$$
$$\mathrm{minWt}((\mathbb{D}_{X_0}(E))_x) = -\,\mathrm{maxWt}(p_*(E)_x) - \mathrm{wt}(\det(\mathfrak{g}^{<0})) \text{ or } = -\infty$$

*Proof.* The second formula follows from the first, using the fact that $p_*$ intertwines the Serre duality functors $\mathbb{D}_X$ on $X$ and $\mathbb{D}_{X_0}$ on $X_0$. $\mathbb{D}_X$ is just linear duality because the isomorphism $T_X \simeq \Omega^1_X$ implies that the canonical bundle $\omega_X \simeq \mathcal{O}_X$. Therefore we will focus on proving the formula for $\mathrm{minWt}(E_x)$.

Note that $p_*(E)_x \simeq p^*(p_*(E))_x$, so we are comparing the fiber weights at $x$ of the two objects $E, p^*(p_*(E)) \in \mathrm{D}^-\mathrm{Coh}^G(X_0)$. We shall factor the closed immersion $p : X_0 \hookrightarrow X$ into two closed immersions

$$\begin{tikzcd}
X_0 \arrow[r, hook, "a"] & X' := \mu^{-1}(\mathfrak{g}^\vee_{\lambda\neq 0}) \arrow[r, hook, "b"] & X \\
& \mathrm{Spec}(k'[\mathfrak{g}_{\lambda=0}[1]]) \arrow[u, "\tilde{x}"] \arrow[r, "\tilde{b}"] & \mathrm{Spec}(k') \arrow[u, "x"'] \\
& \mathrm{Spec}(k') \arrow[u] &
\end{tikzcd} ,$$

where $\mathfrak{g}^\vee_{\lambda\neq 0}$ denotes the sum of eigenspaces of $\mathfrak{g}^\vee$ under $\lambda$ of non-zero weight. The square on the right is cartesian. The subvariety $X' \hookrightarrow X$ is only equivariant for the action of the centralizer $L$ of $\lambda$ in $G$, but this suffices to consider $\lambda$-weights at the fiber $x$. We have $p^*(p_*(E)) \simeq a^*b^*b_*a_*(E)$.

Our first claim is that if $E' \in \mathrm{D}^-\mathrm{Coh}^L(X')$ is such that the weights of $E'_x$ are bounded below, then $\mathrm{minWt}(E'_x) = \mathrm{minWt}(b_*(E')_x)$. By the derived base change formula the complex $b_*(E')_x \simeq \tilde{b}_*\tilde{x}^*(E') \in \mathrm{D}^-\mathrm{Coh}^{\mathbb{G}_m}(\mathrm{Spec}(k'))$. It therefore suffices to show that if $F \in \mathrm{D}^-\mathrm{Coh}^{\mathbb{G}_m}(k[\mathfrak{g}_{\lambda=0}[1]])$ is a complex whose restriction to $\mathrm{Spec}(k')$ has bounded below weights, then $\tilde{b}_*(F)$ has bounded below weights and $\mathrm{minWt}(\tilde{b}_*(F)) = \mathrm{minWt}(F|_x)$, which can be verified immediately by consider a minimal presentation of $F$ and using the fact that the weights of $k[\mathfrak{g}_{\lambda=0}[1]]$ are 0.

To complete the proof of the lemma, it now suffices to show that if $E \in \mathrm{D}^-\mathrm{Coh}^L(X_0)$ is a complex such that the weights of $E_x$ are bounded below, then $a_*(E)_x$ has bounded below weights as well, and

$$\mathrm{minWt}(E_x) = \mathrm{minWt}(a_*(E)_x) - \mathrm{wt}(\det(\mathfrak{g}^{<0})).$$

Because $a$ is a regular embedding, the complex $a^*(a_*(E))$ has a finite filtration whose associated graded is $E \otimes \bigwedge \mathfrak{g}^\vee_{\lambda\neq 0}$, and thus $a_*(E)_x$ has a finite filtration whose associated graded is $E_x \otimes \bigwedge \mathfrak{g}^\vee_{\lambda\neq 0}$. Because we have discarded the 0 weight spaces of $\mathfrak{g}^\vee$, there is a single one dimensional weight space of $\bigwedge \mathfrak{g}^\vee_{\lambda\neq 0}$ whose weight is the minimal weight $\mathrm{wt}(\det(\mathfrak{g}^{<0}))$. It follows that provided the weights of $E_x$ are bounded below, the non-vanishing subcomplex of $a_*(E)_x$ of minimal weight is precisely the minimal weight subcomplex of $E_x$ tensored with $\det(\mathfrak{g}^{<0})$, and the formula follows. $\square$

*Proof of Proposition 3.19.* The proof amounts to comparing the weight windows appearing in the previous two lemmas with the weight windows defining $\mathcal{M}_{\mathcal{X}_0}(\delta)$. Given a point $x \in X_0(k')$ fixed by $\lambda : (\mathbb{G}_m)_{k'} \to G_{k'}$, we have

$$L_{\mathcal{Y}}|_x \simeq [0 \to \Omega^1_{X,x} \oplus \mathfrak{g}^\vee \to \mathfrak{g}^\vee], \text{ whereas}$$
$$L_{\mathcal{X}}|_x \simeq [\mathfrak{g} \to \Omega^1_{X,x} \to \mathfrak{g}^\vee]$$

Self duality implies that $(f^*L_{\mathcal{X}})^{<0} \simeq ((f^*L_{\mathcal{X}})^{>0})^\vee$, so one can compute

$$\eta^{\mathcal{Y}}_f + 2\,\mathrm{wt}(\det(\mathfrak{g}^{<0})) = -a^{\mathcal{X}}_f.$$

So for $F$ as in the statement of the proposition, if the weights of $f^*(\Phi_X(F))$ are bounded below then Lemma 3.21 and Corollary 3.20.1 implies that

$$\begin{aligned}\mathrm{minWt}(\Phi_X(F)_x) &= \mathrm{minWt}(p_*(\Phi_X(F))_x) - \mathrm{wt}(\det(\mathfrak{g}^{<0}))\\ &\geq \mathrm{wt}(f^*\delta) - \frac{1}{2}\eta_f^{\mathcal{Y}} - \mathrm{wt}(\det(\mathfrak{g}^{<0})) = \mathrm{wt}(f^*\delta) + a_f^{\mathcal{X}}.\end{aligned}$$

Similarly if the weights of $\mathbb{D}_{X_0}(\Phi_X(F))$ are bounded below, then

$$\begin{aligned}\mathrm{minWt}(\mathbb{D}_{X_0}(\Phi_X(F))_x) &= -\mathrm{maxWt}(p_*(\Phi_X(F))_x) - \mathrm{wt}(\det(\mathfrak{g}^{<0}))\\ &\geq -\mathrm{wt}(f^*\delta) - \frac{1}{2}\eta_f^{\mathcal{Y}} - \mathrm{wt}(\det(\mathfrak{g}^{<0})) = -\mathrm{wt}(f^*\delta) + a_f^{\mathcal{X}}.\end{aligned}$$

□

## 4. The $D$-equivalence conjecture for moduli spaces of sheaves on a $K3$-surface

The results of the previous section lead to a general statement about derived stacks with self-dual cotangent complex $L_{\mathcal{X}} \simeq (L_{\mathcal{X}})^\vee$. Namely, if $\mathcal{X}$ is such a stack for which $\mathcal{X}^{\mathrm{cl}}$ admits a good moduli space and $\delta \in NS(\mathcal{X})_{\mathbb{R}}$ is *generic* in the sense of **??**, then $\mathcal{X}^{\mathrm{ss}}(\ell)$ is a smooth Deligne-Mumford stack for $\ell \in NS(\mathcal{X})_{\mathbb{R}}$ away from a finite real linear hyperplane arrangement. Furthermore for any $\ell$ for which $\mathcal{X}^{\mathrm{ss}}(\ell)$ is smooth and DM the restriction functor

$$\mathcal{M}_{\mathcal{X}}(\delta) \subset \mathrm{DCoh}(\mathcal{X}) \to \mathrm{DCoh}(\mathcal{X}^{\mathrm{ss}}(\ell))$$

is an equivalence, and hence $\mathcal{X}^{\mathrm{ss}}(\ell)$ for any two generic values of $\ell$ will be derived equivalent. In this section we discuss how this geometric set-up arises in the study of moduli spaces of sheaves on a $K3$-surface. The arguments here are more of a sketch than those appearing in the previous sections, with the full details to appear in forthcoming work.

We will actually consider a slightly more general set up, following the notation of [BM], and we refer the reader there for a more complete discussion: Let $S$ be a $K3$-surface and let $\alpha \in Br(X)$ be a Brauer class. We let $\mathcal{C} = \mathrm{DCoh}(S, \alpha)$ denote the pre-triangulated *dg*-category of $\alpha$-twisted complexes on $S$, and

$$v : K_0(\mathcal{C}) \to H^*_{alg}(X, \alpha, \mathbb{Z})$$

the *Mukai vector* map. Then Bridgeland identifies a connected component $\mathrm{Stab}^\dagger(\mathcal{C})$ of the space of numerical stability conditions.

**Definition 4.1.** For $\sigma \in \mathrm{Stab}^\dagger(\mathcal{C})$ and Mukai vector $v \in H^*_{alg}(S, \alpha, \mathbb{Z})$, we let $\mathcal{M}_\sigma(v)$ denote the moduli stack of $\sigma$-semistable complexes $E \in \mathcal{C}$ in heart of the $t$-structure associated to $\sigma$ with $v(E) = v$.

When $v$ is primitive and $v^2 > 0$ and $\sigma$ is generic for $v$, then the stack $\mathcal{M}_\sigma(v)$ admits a good moduli space which is a smooth projective hyperkähler manifold of dimension $v^2 + 2$. For $\alpha$ trivial and certain stability conditions $\sigma$, $\mathcal{M}_\sigma(v)$ can be identified with the stack of Gieseker semistable coherent sheaves on $S$ of class $v$.

The cotangent complex of $\mathcal{M}_\sigma(v)$ is self-dual, and at a point $[E] \in \mathcal{M}_\sigma(v)$ corresponding to a complex $E \in \mathrm{DCoh}(S, \alpha)$ we have

$$\mathbb{L}_{\mathcal{M}_\sigma(v)}|_{[E]} \simeq \mathrm{RHom}(E, E[1])^\vee.$$

For any family of complexes $E \in \mathrm{DCoh}(S \times B, \alpha)$ over a base $B$, there is a canonical non-trivial map of group schemes $(\mathbb{G}_m)_B \to \mathrm{Aut}_B(E)$ which acts by scaling. This can alternatively be described as a map of sheaves of groups $(\mathbb{G}_m)_{\mathcal{M}_\sigma(v)} \to I_{\mathcal{M}_\sigma(v)}$, where the latter denotes the inertia stack. This implies that the stack $\mathcal{M}_\sigma(v)$ is never Deligne-Mumford, and as a derived stack $\mathcal{M}_\sigma(v)$ is also never smooth, because $\mathbb{L}_{\mathcal{M}_\sigma(v)}$ has a trivial summand of the form $\mathcal{O}_{\mathcal{M}_\sigma(v)}[1]$ which is dual to the lie algebra of this generic $\mathbb{G}_m$ stabilizer. More precisely, we have a decomposition

$$\mathbb{L}_{\mathcal{M}_\sigma(v)} \simeq \mathcal{O}_{\mathcal{M}_\sigma(v)}[1] \oplus (\mathbb{L}_{\mathcal{M}_\sigma(v)})^{\mathrm{rig}} \oplus \mathcal{O}_{\mathcal{M}_\sigma(v)}[-1],$$

where $(\mathbb{L}_{\mathcal{M}_\sigma(v)})^{\mathrm{rig}}$ is again self-dual.

**Proposition 4.2.** *There are derived algebraic stacks $\mathcal{M}_\sigma(v) /\!\!/ \mathbb{G}_m$ and $\mathcal{M}_\sigma^{\mathrm{rig}}(v)$ with maps*

$$\mathcal{M}_\sigma(v) \xrightarrow{\mathbb{G}_m\text{-gerbe}} \mathcal{M}_\sigma(v) /\!\!/ \mathbb{G}_m \xleftarrow[\text{closed immersion}]{\text{surjective}} \mathcal{M}_\sigma^{\mathrm{rig}}(v)$$

*such that the first map rigidifies the generic $\mathbb{G}_m$-stabilizer of $\mathcal{M}_\sigma(v)$, and $\mathbb{L}_{\mathcal{M}_\sigma(v)^{\mathrm{rig}}} \simeq (\mathbb{L}_{\mathcal{M}_\sigma(v)})^{\mathrm{rig}}|_{\mathcal{M}_\sigma(v)^{\mathrm{rig}}}$.*

*Sketch of proof.* In the classical context, the construction of the rigidification $\mathcal{X} /\!\!/ \mathbb{G}_m$ of a stack with an embedding $(\mathbb{G}_m)_{\mathcal{X}} \hookrightarrow I_{\mathcal{X}}$ appears in [AOV]. When $\mathcal{X}$ is a derived stack, we can elevate this to the construction of a derived stack $\mathcal{X} /\!\!/ \mathbb{G}_m$ using deformation theory. The stack $\mathcal{X}$ is canonically obtained as a sequence of square-zero extensions $\mathcal{X}^{\mathrm{cl}} \hookrightarrow \tau_{\leq 1}\mathcal{X} \hookrightarrow \tau_{\leq 2}\mathcal{X} \hookrightarrow \cdots$. Because the map $\mathcal{X}^{\mathrm{cl}} \to \mathcal{X}^{\mathrm{cl}} /\!\!/ \mathbb{G}_m$ is a $\mathbb{G}_m$ gerbe, the pullback functor $\mathrm{D}^-\mathrm{Coh}(\mathcal{X}^{\mathrm{cl}} /\!\!/ \mathbb{G}_m) \to \mathrm{D}^-\mathrm{Coh}(\mathcal{X}^{\mathrm{cl}})$ is $t$-exact and fully faithful and a complex in $\mathrm{D}^-\mathrm{Coh}(\mathcal{X}^{\mathrm{cl}})$ descends to $\mathrm{D}^-\mathrm{Coh}(\mathcal{X}^{\mathrm{cl}}/\mathbb{G}_m)$ if and only if $\mathbb{G}_m$ acts with weight 0 in every fiber. Rutherford, $\mathbb{L}_{\mathcal{X}^{\mathrm{cl}}}$ and $\mathbb{L}_{\mathcal{X}}$ descend to $\mathcal{X}^{\mathrm{cl}}/\mathbb{G}_m$ and applying this reasoning inductively allows one to descend the square-zero extensions

$$\begin{tikzcd}
\mathcal{X}^{\mathrm{cl}} \arrow[r, hook] \arrow[d] & \tau_{\leq 1}\mathcal{X} \arrow[r, hook] \arrow[d] & \tau_{\leq 2}\mathcal{X} \arrow[r, hook] \arrow[d] & \cdots \\
\mathcal{X}^{\mathrm{cl}} /\!\!/ \mathbb{G}_m \arrow[r, hook] & \mathcal{X}_1 \arrow[r, hook] & \mathcal{X}_2 \arrow[r, hook] & \cdots
\end{tikzcd}$$

Then one can define $\mathcal{X} /\!\!/ \mathbb{G}_m := \operatorname{colim}_n \mathcal{X}_n$ and the map $\mathcal{X} \to \mathcal{X} /\!\!/ \mathbb{G}_m$ will be a $\mathbb{G}_m$ gerbe whose inertia completes the short exact sequence $\{1\} \to (\mathbb{G}_m)_{\mathcal{X}} \to I_{\mathcal{X} /\!\!/ \mathbb{G}_m} \to I_{\mathcal{X}}|_{\mathcal{X}} \to \{1\}$.

The the cotangent complex of $\mathcal{M}_\sigma(v) /\!\!/ \mathbb{G}_m$ differs from the cotangent complex of $\mathcal{M}_\sigma(v)$ by a trivial summand in cohomological degree 1, but it still has a trivial summand in cohomological degree $-1$. We will use a trick to remove this extraneous summand in the cotangent complex by considering the determinant map (see [STV])

$$\det : \mathcal{M}_\sigma(v) \to \operatorname{Pic}(S)_\omega$$

where $\operatorname{Pic}(S)$ is the derived Picard stack of $S$ of invertible sheaves of numerical class $\omega \in NS(S)$. Note that the class $\omega$ is determined by the Mukai vector $v$. Because $H^1(S, \mathcal{O}_S) = 0$, we have $\operatorname{Pic}(S)_\omega \simeq \operatorname{Spec}(k[\epsilon[1]])/\mathbb{G}_m$, where $\mathbb{G}_m$ acts trivially on $k[\epsilon[1]]$. The determinant map descends to a map $\mathcal{M}_\sigma(v) /\!\!/ \mathbb{G}_m \to \operatorname{Pic}(S)_\omega /\!\!/ \mathbb{G}_m = \operatorname{Spec}(k[\epsilon[1]])$, and we define

$$\mathcal{M}_\sigma(v)^{\mathrm{rig}} := \mathcal{M}_\sigma(v) \times_{\operatorname{Spec}(k[\epsilon[1]])} \operatorname{Spec}(k).$$

Technically this only works when the complexes of class $v$ have non-zero rank, but one can use the trick of finding a derived equivalence or anti-equivalence with a different $K3$ surface which identifies $\mathcal{M}_\sigma(v)$ with a moduli space of complexes of non-zero rank.

□

Although it was shown in [BM] that $\mathcal{M}_\sigma(v)^{\mathrm{cl}}$ admits a good moduli space when $\sigma$ is generic for $v$, we will need to consider good moduli spaces when $\sigma$ is *arbitrary*. This will appear in forthcoming joint work with the author, Jarod Alper, and Jochen Heinloth as part of a larger study on the construction and properties of good moduli spaces in the theory of $\Theta$-stability:

**Theorem 4.3.** [AHR] *The stack* $\mathcal{M}^{\mathrm{rig}}_{\sigma}(v)^{\mathrm{cl}}$ *admits a good moduli space for any* $\sigma \in \mathrm{Stab}^{\dagger}(\mathcal{C})$ *and primitive* $v \in H^*_{alg}(S,\alpha,\mathbb{Z})$ *with* $v^2+2>0$.

Now given a $\sigma_0 \in \mathrm{Stab}^{\dagger}(\mathcal{C})$, one has a canonical map from $(\mathbb{Z}v)^{\perp} \subset H^*_{alg}(S,\alpha,\mathbb{Z}) \simeq K_0^{num}(S,\alpha)$ to $NS(\mathcal{M}^{\mathrm{rig}}_{\sigma_0}(v))$, inducing a map

$$(\mathbb{R}\cdot v)^{\perp} \to NS(\mathcal{M}^{\mathrm{rig}}_{\sigma_0}(v))$$

Furthermore, in [BM, Section 10] they describe a map

$$\ell : \mathrm{Stab}^{\dagger}(\mathcal{C}) \to (\mathbb{R}\cdot v)^{\perp} \to NS(\mathcal{M}^{\mathrm{rig}}_{\sigma_0}(v))$$

under which $\sigma_0$ maps to a Neron-Severi class which descends to an an ample bundle on the good moduli space of $\mathcal{M}^{\mathrm{rig}}_{\sigma_0}(v)$. Furthermore semistability for $\sigma \in \mathrm{Stab}^{\dagger}(\mathcal{C})$ which arise as small perturbations of $\sigma_0$ can be identified with $\Theta$-semistability on the stack $\mathcal{M}^{\mathrm{rig}}_{\sigma_0}(v)$ with respect to $\ell(\sigma)$.

In light of this observation we can apply Theorem 4.3 and Theorem 3.6 in this particular situation to obtain:

**Theorem 4.4.** *Let* $\sigma_0 \in \mathrm{Stab}^{\dagger}(\mathcal{C})$ *and let* $\sigma$ *be a small perturbation of* $\sigma_0$ *such that* $\mathcal{M}^{\mathrm{rig}}_{\sigma}(v) \subset \mathcal{M}^{\mathrm{rig}}_{\sigma_0}(v)$ *and* $\sigma$ *is generic for* $v$. *Then for any* $\delta \in NS(\mathcal{M}^{\mathrm{rig}}_{\sigma_0}(v))_{\mathbb{R}}$ *which is* ***generic****, the restriction functor induces an equivalence of derived categories*

$$\begin{array}{ccc} \mathcal{M}_{\mathcal{M}^{\mathrm{rig}}_{\sigma_0}(v)}(\delta) & \xrightarrow[res]{\simeq} & \mathrm{DCoh}(\mathcal{M}^{\mathrm{rig}}_{\sigma}(v)) \\ \cap & & \\ \mathrm{DCoh}(\mathcal{M}^{\mathrm{rig}}_{\sigma_0}(v)) & & \end{array}$$

The last thing which needs to be checked is that there exists a generic $\delta \in NS(\mathcal{M}^{\mathrm{rig}}_{\sigma_0}(v))_{\mathbb{R}}$. As remarked above it suffices to test the condition of genericity only at closed points of $\mathcal{M}^{\mathrm{rig}}_{\sigma_0}(v)$, i.e. points which correspond to polystable objects $E \in \mathcal{C}$. We write $E = \bigoplus E_i \otimes V_i$ where $E_i$ are non-isomorphic simple objects of $\mathcal{C}$, and the $V_i$ are certain multiplicity vector spaces. Then

$$\mathrm{Aut}(E) = \prod_i GL(V_i)/\{(\mathbb{C}^{\times}) \cdot \prod_i \mathbf{1}_{V_i}\}, \text{ and}$$

$$H^0(T_{[E]}\mathcal{M}^{\mathrm{rig}}_{\sigma_0}(v)) = H^0(T_{[E]}\mathcal{M}_{\sigma_0}(v)) \simeq \bigoplus_{i,j} \mathrm{Hom}(E_i, E_j[1]) \otimes \mathrm{Hom}(V_i, V_j).$$

As Serre duality implies $\mathrm{Hom}(E_i, E_j[1]) \simeq \mathrm{Hom}(E_j, E_i[1])^{\vee}$, we can identify the space $H^0(T_{[E]}\mathcal{M}^{\mathrm{rig}}_{\sigma_0}(v))$ with the cotangent space of the space of representations of a quiver with: one vertex for each index $i$, and $\dim(\mathrm{Hom}(E_i, E_j[1]))$ arrows from vertex $i$ to vertex $j$ for $i \le j$.

The analysis of genericity for representations of this kind was carried out in [HLS, Section 5.1]. The analysis there holds in our situation with only one slight modification – there the automorphism group did not involve the quotient by scalar matrices, which introduced the hypothesis that the quiver have a non-trivial framing vertex. That is no longer necessary here, so the argument shows that there is some $\delta \in NS(\mathcal{M}^{\mathrm{rig}}_{\sigma_0}(v))_{\mathbb{R}}$ which is generic for the point $[E]$. As $E$ varies over all polystable objects, there will only be a locally finite collection of real hypersurfaces along which $\delta \in NS(\mathcal{M}^{\mathrm{rig}}_{\sigma_0}(v))_{\mathbb{R}}$ is non-generic at $[E]$, hence there will exist $\delta$ which are generic everywhere. We can thus conclude.

**Corollary 4.4.1.** *For a flop* $\mathcal{M}^{\mathrm{rig}}_{\sigma_+}(v) \dashrightarrow \mathcal{M}^{\mathrm{rig}}_{\sigma_-}(v)$ *induced by perturbing a stability condition* $\sigma_0 \in \mathrm{Stab}^{\dagger}(\mathcal{C})$ *which lies on a wall to two* $v$*-generic stability conditions* $\sigma_{\pm}$*, the equivalences of Theorem 4.4 give a derived equivalence*

$$\mathrm{DCoh}(\mathcal{M}^{\mathrm{rig}}_{\sigma_+}(v)) \simeq \mathrm{DCoh}(\mathcal{M}^{\mathrm{rig}}_{\sigma_-}(v)).$$

One can deduce theorem 0.2 from the detailed analysis of the minimal model program from moduli spaces $\mathcal{M}^{\mathrm{rig}}_\sigma(v)$ completed in [BM], namely: 1) any Calabi-Yau manifold which is birational to $\mathcal{M}_\sigma(v)$ for some twisted $K3$ surface $(S,\alpha)$ and some $v \in H^*_{alg}(S,\alpha,\mathbb{Z})$ is in fact isomorphic to such a moduli space (possibly for a different $(S',\alpha')$ with an equivalent derived category), and 2) any birational equivalence between two such Calabi-Yau manifolds can be realized as a sequence of flops of the kind appearing in Corollary 4.4.1.

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