Moduli spaces of objects in dg-categories

Miami, Feb. 1, 2025

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Goals of the talk

1. Discuss framework for studying moduli problems in algebraic geometry
2. Apply this framework to moduli of objects in dg-categories
3. Discuss implications for homological mirror symmetry

Motivation

Homological mirror symmetry: For a Calabi-Yau manifold M, there is a mirror CY $M^\vee$ such that

$$\mathrm{D}^b\mathrm{Fuk}(M) \cong \mathrm{D}^b\mathrm{Coh}(M^\vee).$$

Proposed constructions of $M^\vee$

1. Strominger-Yau-Zaslow (1996): $M^\vee =$ moduli space of special Lagrangian tori in M ← developed
2. Pomerleano, Gross-Siebert (∼2015, [3, 6]): coordinate ring of $M^\vee$ determined by log GW invariants of M or SH^0 ← developed
3. Toën-Vaquié (2015, [8]): $M^\vee =$ moduli space of objects in $\mathrm{D}^b\mathrm{Fuk}(M)$ ← we will develop this today

Challenges: (1) constructing special Lagrangian fibrations and compactifying $M^\vee$, (2) resolving singularities of $M^\vee$

Non-commutative algebraic geometry

Ground field k, $\mathrm{char}(k) = 0$.

- (Kontsevich-Soibelman, [5]) A non-commutative space is a smooth and proper dg-category $\mathscr{C}$
- (Toën-Vaquié, [7]) $\mathscr{C} \cong \mathrm{Perf}(A^\bullet)$ for $A^\bullet$ a dg-algebra with $\dim H^*(A^\bullet) < \infty$ and $A^\bullet \in \mathrm{Perf}(A \otimes A^{\mathrm{op}})$.
  $\mathrm{Perf}(A) \subset \mathrm{Mod}_{dg}(A)$ is the smallest thick triangulated subcategory containing A.

Examples

- $\mathrm{D}^b\mathrm{Coh}(X)$, with X a smooth and proper variety
- $\mathrm{D}^b\,\mathrm{Mod}_{fin}(A)$, with A a finite dimensional algebra of finite global dimensional
- $\mathrm{D}^b\mathrm{Fuk}(M)$, with M a compact symplectic manifold with generating collection of Lagrangians

The moduli problem

1. A *stack* is a functor $\mathscr{X} : k\text{—Alg} \to \text{Groupoid}$, + descent
2. A *derived stack* is a functor $\mathscr{X}$: $\text{CDGA}^{\leq 0}_{/k} \to \infty - \text{Groupoid}$
3. A derived stack is *n-geometric and f.p.* if there is a cover $U = \mathrm{Spec}(R) \to \mathscr{X}$ with $R \in CDGA^{\leq 0}$ of finite presentation over k such that $U \times_{\mathscr{X}} U$ is $(n-1)$-geometric and f.p. (0-geometric if $\mathscr{X} \cong \mathrm{Spec}(R)$.)

For R either a k-algebra or non-positively graded $CDGA$ over k, define

$$\mathscr{M}(R) = \mathrm{Perf}(R \otimes_k A^\bullet)^{\cong}$$

$(-)^{\cong}$ is the subcategory of isomorphisms.

Theorem (Toën-Vaquié, 2005, [7])

$\mathscr{M}$ is a union of finitely presented n-geometric open substacks.

Key concept: t-structures

Definition (t-structure)

A t-structure on $\mathscr{C}$ consists of subcategories $(\mathscr{C}_{>0}, \mathscr{C}_{\leq 0})$ satisfying:

- $\mathscr{C}_{>0}[1] \subset \mathscr{C}_{>0}$,
- $\mathrm{Hom}(\mathscr{C}_{>0}, \mathscr{C}_{\leq 0}) = 0$, and
- $\forall E \in \mathscr{C}$, $\exists$ an exact triangle $E_{>0} \to E \to E_{\leq 0} \to$, with $E_{>0} \in \mathscr{C}_{>0}$ and $E_{\leq 0} \in \mathscr{C}_{\leq 0}$.

Definition (The heart)

$\mathscr{C}^{\heartsuit} := \mathscr{C}_{>-1} \cap \mathscr{C}_{\leq 0} \subset \mathscr{C}$. It is an abelian category.

Example: In the standard t-structure on $\mathrm{D}^b\mathrm{Mod}(R)$, $\mathscr{C}_{>0}$ consists of complexes with $H_i(F_\bullet) \cong 0$ for $i \leq 0$.

A more manageable moduli stack

Equip $\mathscr{C} = \mathrm{Perf}(A)$ with a t-structure.

$\forall R \in CDGA^{\leq 0}_{/k}$, this induces a t-structure on $\mathrm{Mod}_{dg}(R \otimes_k A)$:

- $\mathrm{Mod}_{dg}(R \otimes_k A)_{>0}$ generated under extensions and colimits by $R \otimes E$ with $E \in \mathscr{C}_{>0}$.

We can identify a *classical* substack:

$$\mathscr{M}^{\heartsuit}(R) := \left\{ E \in \mathscr{M}(R) \mid S \otimes_R E \in \mathrm{Mod}_{dg}(S \otimes A)^{\heartsuit}, \forall R \to S \right\}$$

(For a k-algebra R, $\mathscr{M}^{\heartsuit}(R)$ is a groupoid, not an ∞-groupoid)

Example

- $\mathscr{M}^{\heartsuit}$ = flat families of coherent sheaves on a variety X
- $\mathscr{M}^{\heartsuit} = \bigsqcup_{n \geq 0} \mathrm{Rep}_n(A)/\mathrm{GL}_n \subset \mathscr{M}_{\mathrm{D}^b \mathrm{Mod}_{fin}(A)}$

Moduli spaces in algebraic geometry

Suppose $\mathscr{X}$ is a 1-geometric f.p. stack.

Definition (Alper, 2008, [1])

A good moduli space $q : \mathscr{X} \to X$ is a morphism to an algebraic space s.t. $q_* : \mathrm{QCoh}(\mathscr{X}) \to \mathrm{QCoh}(X)$ exact and $q_*(\mathscr{O}_{\mathscr{X}}) = \mathscr{O}_X$.

- Consequence: q is the universal map to an algebraic space.
- Étale locally over X, it looks like a GIT quotient morphism $\mathrm{Spec}(R)/G \to \mathrm{Spec}(R^G)$ (Alper-Hall-Rydh, 2019).
- Since (Alper-HL-Heinloth,2018), we have effective ways to construct moduli spaces directly, without GIT

Θ-semistability

If $\mathscr{X}$ is only *locally* f.p., you need a notion of semistability. A *numerical invariant* is a locally constant $\nu : \mathrm{Map}(\Theta, \mathscr{X}) \to \mathbb{R}$, where $\Theta := \mathbb{A}^1/\mathbb{G}_m$ (some hypotheses omitted).

Example: For the stack of vector bundles on a Riemann surface, a map $f : \Theta \to \mathscr{X}$ is a filtration of bundles $\cdots V_{w+1} \subset V_w \subset \cdots$. If $G_w := V_w/V_{w+1}$, then

$$\nu(f) := \sum_w w(\frac{D}{R}\mathrm{rank}(G_w) - \deg(G_w)) + \sum_w w^2 \mathrm{rank}(G_w)$$

Definition (Θ-semistability, [4])

A point $x \in \mathscr{X}$ is semistable if for all $f : \Theta \to \mathscr{X}$ with $f(1) \cong x$, $\nu(f) \geq 0$. An HN filtration is an f that minimizes ν.*

*Should allow $\mathbb{Q}$-filtrations or $\mathbb{R}$-filtrations in general.

A general structure theorem

Definition (Codimension two filling conditions)

$\mathscr{X}$ is Θ- *and S-complete* if for any regular surface S with non-trivial $\mathbb{G}_m$-action and closed fixed point $0 \in S$, any $\mathbb{G}_m$-equivariant family over $S \setminus 0$ extends equivariantly and uniquely to S.

Suppose $\mathscr{X}$ is locally f.p., 1-geometric, Θ- and S-complete, and families over punctured discs extend over punctures.

Theorem ([2, 4])

1. HN-boundedness $\Rightarrow$ every $x \in \mathscr{X}$ has a unique HN-filtration, and these give a locally closed stratification of $\mathscr{X}$ (a Θ-stratification), with $\mathscr{X}^{\mathrm{ss}} \subset \mathscr{X}$ open
2. $\mathscr{X}^{\mathrm{ss}}$ bounded $\Rightarrow$ $\mathscr{X}^{\mathrm{ss}}$ has a proper good moduli space

Stability conditions

Fix a free abelian group Λ of finite rank with norm $\|-\|$ and a locally constant additive map $v : \mathscr{M} \to \Lambda$.

Example: For $\mathrm{D}^b\mathrm{Coh}(X)$, v is the Chern character and $\Lambda \subset H^*(X;\mathbb{Q})$ its image.

A stability condition on $\mathscr{C}$ consists of:

1. t-structures $(\mathscr{C}_{>\varphi}, \mathscr{C}_{\leq\varphi})_{\varphi\in\mathbb{R}}$
2. homomorphism $Z : \Lambda \to \mathbb{C}$

We require $\exists c \in \mathbb{R}, w \in (0,1)$ such that:

(a) $\mathscr{C}_{>\varphi} = \bigcup_{a>\varphi} \mathscr{C}_{>a}$

(b) $\forall E \in \mathscr{C}$, $E \in \mathscr{C}_{(a,b]} := \mathscr{C}_{>a} \cap \mathscr{C}_{\leq b}$ for some $a < b$

(c) If $b - a = w$ and $0 \neq E \in \mathscr{C}_{(a,b]}$ with $v = v(E)$, then

$$Z(v)/\|v\| \in e^{[c,\infty)+i\pi(a,b]}$$

Stability conditions II

This is not the original definition of a stability condition, but it is equivalent:

1. We can define $\phi^+(E) = \inf\{b | E \in \mathscr{C}_{\leq b}\}$ and $\phi^-(E) = \sup\{a | E \in \mathscr{C}_{>a}\}$
2. E is *semistable* of phase ϕ if $\phi^+(E) = \phi^-(E) = \phi$
3. Every $E \in \mathscr{C}$ has a HN filtration $E_1 \to E_2 \to \cdots$ with $\mathrm{gr}_i(E_\bullet)$ semistable of phase ϕ_i, and $\phi_1 > \phi_2 > \cdots > \phi_N$.
4. For $E \in \mathscr{C}$, define the mass $m(E) = \sum_i |Z(\mathrm{gr}_i(E))|$

For experts: We call the structure $(\mathscr{C}_{>\varphi}, \mathscr{C}_{\leq\varphi})_{\varphi\in\mathbb{R}}$ a sluicing. (Also require an "almost noetherian" hypothesis.) If $R \to S$ is a composition of polynomial algebras, localizations, and perfect algebras, then a sluicing on $\mathrm{Perf}(R \otimes A)$ induces one on $\mathrm{Perf}(S \otimes A)$.

Main theorem

Mass-Hom inequality for a stability condition: $\exists C > 0$ such that $\forall M^\bullet \in \mathrm{Perf}(A^\bullet)$,

$$\dim(H^*(M^\bullet)) < Cm(M^\bullet).$$

Theorem (HL-Robotis)

If mass-Hom inequality holds, then $\forall \varphi \in \mathbb{R}$, letting $\mathscr{M}^\varphi := \mathscr{M}^\heartsuit$ for the t-structure with $\mathscr{C}^\heartsuit = \mathscr{C}_{(\varphi-1,\varphi]}$,

1. $\mathscr{M}^\varphi \subset \mathscr{M}$ is a 1-geometric open substack,
2. $\mathscr{M}^\varphi$ has a Θ-stratification by HN filtrations, and $\forall M > 0$, $\{E | m(E) < M\} \subset \mathscr{M}^\varphi$ is a bounded open union of strata, and
3. $\mathscr{M}_v^{\mathrm{ss}} \subset \mathscr{M}^\varphi$ is open and has a proper good moduli space, $\forall v$.

(Condition (3) implies $\dim H^*(M^\bullet) \leq f(m(M^\bullet))$ for some $f(-)$.)

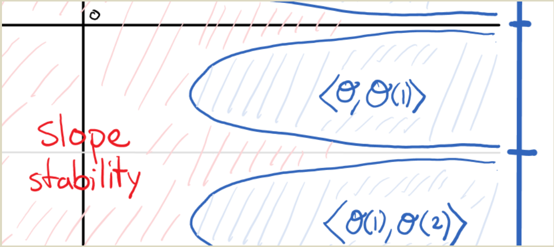
The key idea

Boundedness! $\mathscr{M}^{\heartsuit} \subset \mathscr{M}$ is automatically Θ- and S-complete, so general theorem applies.

- HN-boundedness amounts to this claim: for a bounded set $S \subset \mathscr{M}$, the set $\{\mathrm{gr}_i^{\mathrm{HN}}(E) | E \in S\}$ is also bounded
- Toën-Vaquié show that the forgetful functor f : $\mathscr{M} \to \underline{\mathrm{Perf}}$ – taking $M^\bullet \in \mathrm{Perf}(R \otimes A^\bullet)$ and regarding it as an object in $\mathrm{Perf}(R)$ – is quasi-compact
- The set $\{E \in \mathscr{C}_{(\varphi-1,\varphi]} | m(E) < M\}$ is contained in the preimage of a bounded subset of $\underline{\mathrm{Perf}}$, hence bounded
- This simultaneously implies HN-boundedness and boundedness of $\mathscr{M}_v^{\mathrm{ss}}$

Motivation: the space $\mathrm{Astab}(\mathscr{C})$

1. Bridgeland's theorem: $\exists$ a metric topology on $\mathrm{Stab}(\mathscr{C})$ such that the forgetful map $\mathrm{Stab}(\mathscr{C}) \to \mathrm{Hom}(\Lambda, \mathbb{C})$ is a local homeomorphism
2. With Robotis: we construct a partial compactification $\mathrm{Stab}(\mathscr{C})/\mathbb{C} \subset \mathrm{Astab}(\mathscr{C})$, the space of augmented stability conditions, and conjecture it is a manifold with corners
3. Admissible boundary points have connected neighborhoods for any $\mathscr{C}$ $\Rightarrow$ mass-Hom bound for any $\mathscr{C}$.



Additional comments

- We conjecture the mass-Hom bound holds for any stability condition on $\mathscr{C}$ – otherwise, maybe that should be added to the definition
- Mass-Hom inequality was previously considered by Ikeda and Fan-Filip-Haiden-Katzarkov-Liu
- For $\mathscr{C} = \mathrm{D}^b\mathrm{Coh}(C)$ for a compact Riemann surface C, mass-Hom bound corresponds to Le Potier's upper bounds on $\dim H^0(C,F)$ for a semistable vector bundle F
- Mass-Hom inequality only depends on the path component of $\sigma \in \mathrm{Stab}(\mathscr{C})$, holds for any σ such that $\mathscr{C}_{(\varphi-1,\varphi]}$ is artinian, and is preserved by gluing along semiorthogonal decompositions.
- E.g., the theorem gives a new construction of moduli spaces of Gieseker semistable sheaves on $\mathbb{P}^2$.

Assume the mass-Hom inequality, and that $\mathscr{C}$ is 3CY, meaning $\mathrm{Hom}(E,F) \cong \mathrm{Hom}(F,E[3])^*$, and connected.

Lemma (For fixed $v \in \Lambda$ and $\varphi \in \mathbb{R}$)

If $\forall E,F \in \mathscr{M}_v^{ss}$, $\dim \mathrm{Hom}(E,E) = 1$, $\dim \mathrm{Ext}^1(E,F) = 3$ if $E \cong F$ and 0 otherwise, then the moduli space $M_v^\vee$ is a compact Calabi-Yau 3-manifold, and $\mathsf{E}_{\mathrm{univ}} \in \mathscr{C} \otimes \mathrm{Perf}(\mathscr{M}_v^{ss})$ defines an equivalence

$$R\Gamma(\mathscr{M}_v^{ss}, \mathsf{E}_{\mathrm{univ}} \otimes (-)) : \mathrm{Perf}^\tau(M_v^\vee) \xrightarrow{\cong} \mathscr{C}.$$

Here τ denotes a twist by the Brauer class opposite that of the $\mathbb{G}_m$-gerbe $\mathscr{M}_v^{ss} \to M_v^\vee$.

A mystery: if $\mathscr{C}$ is CY3, virtual dimension of $\mathscr{M}_v^{ss}$ is 0. These moduli spaces have no right to exist!

Translation into symplectic topology

Let (M, J, g, Ω) be a Calabi-Yau manifold of complex dimension n. (J complex structure, g Kaëhler metric, $\Omega \in H^0(M, \Omega^2_M)$ holomorphic volume form)

Objects of $\mathrm{D}^b\mathrm{Fuk}(M)$ are compact "immersed Lagrangian branes" $f : L \to M$. In addition to an orientation, a local system E on L, relative spin structure, and a bounding cochain, L has a grading function $\theta_L : L \to \mathbb{R}$ such that $\Omega|_L = e^{i\theta_L} d\mathrm{Vol}_L$.

Say L is w-almost calibrated if θ_L takes values in some interval of width $< w$.

We will always assume M has a generating set of Lagrangians $L_1, \ldots, L_n$ satisfying Abouzaid's criterion.

Some conjectures

The Thomas-Yau-Joyce proposal [?TS, ?joyce]: there is a stability condition on $\mathscr{C} = \mathrm{D}^b\mathrm{Fuk}(M)$ with

- $\mathscr{C}_{(a,b]} \subset \mathscr{C}$ are the objects represented by Lagrangian branes with $\mathrm{im}(\theta_L) \subset (a,b]$ (Including a or $b = \infty$), and $Z(L) = \int_L \Omega$.

Quantitative conjectures arise from this proposal, for $0 < w < 1$, fixed Lagrangian brane L', fixed norm $\|\bullet\|$ on $H_n(M)$:

Conjecture

$\exists C > 0$ such that for any w-almost calibrated L,

- $\|[L]\| \leq C \cdot \mathrm{Vol}(L)$, and $\leftarrow$ support property
- $\dim HF^0(L', L) \leq C \cdot \mathrm{Vol}(L)$. $\leftarrow$ mass-Hom bound

The existence of HN filtrations is usually equated with the convergence of Lagrangian mean curvature flow (with surgery), but this is not necessary:

Lemma

If the conjectured inequalities hold, and $\exists \varphi_1, \varphi_2 \in \mathbb{R}$ with $\varphi_1 - \varphi_2 \notin \mathbb{Z}$ such that any $L \in \mathrm{D}^b\mathrm{Fuk}(M)$ lies in an exact triangle

$$L_1 \to L \to L_2 \to$$

with $\theta_{L_1} > \pi\varphi_i$ and $\theta_{L_2} \leq \pi\varphi_i$, then the proposal defines a stability condition on $\mathrm{D}^b\mathrm{Fuk}(M)$ with a mass-Hom bound.

Question: Can one verify these conditions without showing convergence of mean curvature flow?

What is homological mirror symmetry?

Start with a Calabi-Yau manifold M. Original proposal: $\exists$ a mirror manifold $M^\vee$ and $\exists$ equivalence $\mathrm{D}^b\mathrm{Fuk}(M) \cong \mathrm{D}^b\mathrm{Coh}(M^\vee)$.

The updated proposal: (Partially inspired by [6])

1. Verify the inequalities $\|[L]\| \leq C\mathrm{Vol}(L)$ and $\dim HF^0(L',L) \leq C\mathrm{Vol}(L)$ for L w-almost calibrated
2. Show that every L is an extension $L_1 \to L \to L_2 \to$, as in the lemma
3. Find an embedded special Lagrangian $L \cong T^n$, with $v = [L] \in H_n(M)$, e.g., with the methods of Yang Li. Perhaps L being w-almost calibrated is enough.
4. This gives a functor $\Phi : \mathrm{Perf}^\tau(M_v^{ss}) \to \mathscr{C}$.

Now one can ask, is Φ fully-faithful? (If M_v^{ss} were also smooth this would imply Φ is an equivalence.)

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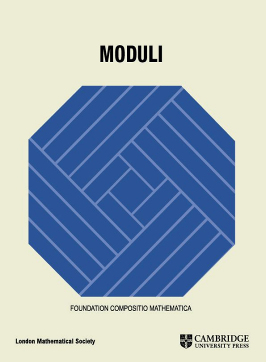

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